On blowing-ups of the projective plane

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Let X be a normal $\mathbb{Q}$-factorial projective algebraic variety.

- Denote by $\mathrm{CDiv}(X)$ the group of Cartier divisors of X. Two such divisors D_1, D_2 are *numerically equivalent*, denoted by $D_1 \equiv D_2$, if $D_1 \cdot C = D_2 \cdot C$ for any curve C of X.
- Let $\mathrm{Num}(X)$ be the subgroup of $\mathrm{CDiv}(X)$ consisting of divisors numerically equivalent to zero. The *Néron–Severi group* of X is:

$$\mathrm{N}^1(X) := \mathrm{CDiv}(X)/\,\mathrm{Num}(X).$$

The tensor product of this abelian group with the real numbers is denoted by $\mathrm{N}^1(X)_{\mathbb{R}}$, while its dual with respect to the intersection form is $\mathrm{N}_1(X)_{\mathbb{R}}$.

Via the perfect duality $N^1(X)_{\mathbb{R}} \times N_1(X)_{\mathbb{R}} \to \mathbb{R}$ the following cones are dual to each other:

Divisors	Curves
$\overline{\mathrm{Eff}}(X)$	$\overline{\mathrm{Mov}}(X)$
$\mathrm{Nef}(X)$	$\overline{\mathrm{NE}}(X)$

where $\mathrm{Eff}(X)$ is generated by the classes of effective divisors and $\mathrm{NE}(X)$ is generated by the classes of curves.

Theorem (Cone theorem). Assume that X is smooth.

1. There are countably many rational curves $C_i \subseteq X$, with $0 < -C_i \cdot K_X \leq \dim X + 1$, such that

$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)_{K_X \geq 0} + \sum_i \mathbb{R}_{\geq 0} \cdot [C_i].$$

2. Given an ample divisor H and an $\varepsilon > 0$ there are only finitely many C_i such that $C_i \cdot (K_X + \varepsilon H) \leq 0$.

Observation. If X is a surface then the following hold.

- $N^1(X)_{\mathbb{R}} = N_1(X)_{\mathbb{R}}$ so that $\overline{\mathrm{Eff}}(X) = \overline{\mathrm{NE}}(X)$ and $\mathrm{Nef}(X) = \overline{\mathrm{Mov}}(X)$. In this case the pseudoeffective and nef cones are dual to each other.
- By the Riemann-Roch theorem the positive light cone

  $$Q := \mathrm{cone}([D] \, : \, D^2 \geq 0 \text{ and } D \cdot H \geq 0 \text{ with } H \text{ ample}\}$$

  is contained in $\overline{\mathrm{Eff}}(X)$. If X has Picard number at least three then Q is circular.

Theorem. Let X_r be the blow-up of the projective plane at r points in very general position. Then $\overline{\mathrm{Eff}}(X_r)$ is polyhedral (i.e. finitely generated) if and only if $0 \leq r \leq 8$.

Idea of proof.

Let H be the pullback of a line and let $E_1, \dots, E_r$ be the exceptional divisors.

- If $0 \leq r \leq 8$ then $-K_{X_r}$ is ample and one concludes by the Cone theorem.
- If on the other hand $r \geq 9$ then $D \sim 3H - E_1 - \cdots - E_9$ generates an extremal ray of $\mathrm{Eff}(X_r)$, because $\mathcal{O}_D(D)$ is non-torsion. Since $[D] \in \partial Q$ and Q is circular it follows that $\overline{\mathrm{Eff}}(X_r)$ cannot be polyhedral.

□

Conjectures.

1. (SHGH). If D is effective with $D \cdot E \geq -1$ for any (-1)-curve E then $h^1(X_r, D) = 0$.
2. The only negative curves of X_r are (-1)-curves.
3. (Nagata). The class of $H - \frac{1}{\sqrt{r}} \sum_{i=1}^{r} E_i$ is nef.

The following implications hold: $(1) \Rightarrow (2) \Rightarrow (3)$. Nagata conjecture holds when r is a square and a version of Nagata conjecture for symplectic varieties has been proved by Biran [1].

To prove (2) $\Rightarrow$ (3) for $r \geq 9$ assume $C \sim dH - \sum_{i=1}^{r} m_i E_i$ is effective and irreducible.

- If $C^2 \geq 0$ then by Cauchy–Schwarz inequality we have

$$d^2 - \frac{1}{r}\left(\sum_{i=1}^{r} m_i\right)^2 \geq d^2 - \sum_{i=1}^{r} m_i^2 = C^2 \geq 0.$$

- If $C^2 < 0$, then C is a (-1)-curve so that

$$\sqrt{r}\, d - \sum_{i=1}^{r} m_i \geq 3d - \sum_{i=1}^{r} m_i = C \cdot (-K) = 1.$$

In both cases $H - \frac{1}{\sqrt{r}} \sum_{i=1}^{r} E_i$ is nef.

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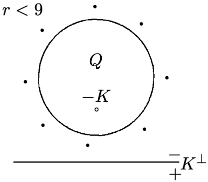
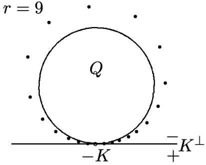
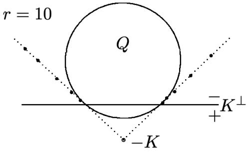
Surfaces

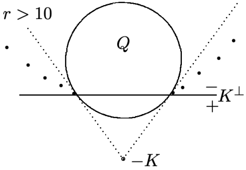
The first two of the following pictures, taken from [3], show the structure of the effective cone for $r \leq 9$.





The second two pictures show that for $r > 10$ the inclusion $Q_{K \geq 0} \subsetneq \overline{\mathrm{Eff}}(X_r)_{K \geq 0}$ is strict.





Definition. Let $x \in X$, let D be a nef divisor of X and let $\pi\colon X' := \mathrm{Bl}_x(X) \to X$ be the blow-up with exceptional divisor E. The *Seshadri constant* of D at x is:

$$\varepsilon(D; x) := \max\{\varepsilon \geq 0 \,:\, \pi^*D - \varepsilon E \text{ is nef}\}.$$

Observation. It is possible to show that

$$\varepsilon(D; x) = \inf_{x \in C \subseteq X} \frac{D \cdot C}{\mathrm{mult}_x C}.$$

Theorem ([4]). Let $r \geq 9$. Then either X_r contains an ample line bundle with irrational Seshadri constant or the SHGH conjecture fails on X_{r+1}.

Idea of proof.

Assume any ample divisor of X_r has a rational Seshadri constant and the SHGH holds for X_r and X_{r+1}. Let $E = E_1 + \cdots + E_r$.

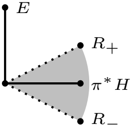
- By Nagata $D \sim H - \alpha E$ is ample for $\frac{1}{\sqrt{r+1}} < \alpha < \frac{1}{\sqrt{r}}$.
- If the Seshadri constant ε of D at a very general $p \in X_r$ is rational, then there is a curve C of the form $dH - mE$ which computes it. Let $\tilde{C} \sim \pi^* C - \mu E_p$ be its strict transform.
- From $\varepsilon = \frac{D \cdot C}{\mu} < \sqrt{D^2}$ and $\tilde{C} \cdot (H - \frac{1}{\sqrt{r+1}} E) \geq 0$ one deduces

$$d \geq 2m + \mu.$$

- This inequality, together with the SHGH conjecture, imply that the expected dimension of the strict transform of C is non-negative, but the above inequalities imply that it is negative, a contradiction.

Theorem. Let a, b, c be pairwise coprime positive integers, let $\mathbb{P}(a, b, c)$ be the corresponding weighted projective plane, let $p = (1, 1, 1)$ and let H be a divisor of degree 1. If $\varepsilon(H; p) = \frac{1}{\sqrt{abc}}$ then the Nagata conjecture holds for $r = abc$.

Ideal of proof. Let $\pi\colon X \to \mathbb{P}(a,b,c)$ be the the blowing-up at p with exceptional divisor E. In the following picture the shaded region is the positive light cone Q with extremal rays generated by $R_{\pm} = \pi^*H \pm \frac{1}{\sqrt{abc}}E$.



If $\varepsilon(H;p) = \frac{1}{\sqrt{abc}}$ then R_- is nef.

Let

$$f\colon \mathbb{P}^2 \to \mathbb{P}(a,b,c), \qquad (x,y,z) \mapsto (x^a, y^b, z^c)$$

and let Y_r be the blowing-up of $\mathbb{P}^2$ at the $r = abc$ points of $f^{-1}(p)$. Since f^*R_- is nef on Y_r the same class is nef on X_r because

$$\overline{\mathrm{Eff}}(X_r) \subseteq \overline{\mathrm{Eff}}(Y_r)$$

by the semicontinuity of the dimension of cohomology groups in families. □

References

[1] Paul Biran. From symplectic packing to algebraic geometry and back. European Congress of Mathematics, Vol. II (Barcelona, 2000). Progr. Math., vol. 202. Birkhäuser, Basel., pages 507–524. 2001.

[2] Steven Dale Cutkosky and Kazuhiko Kurano. Asymptotic regularity of powers of ideals of points in a weighted projective plane. *Kyoto J. Math.* 51 (1):25–45, 2011.

[3] Tommaso de Fernex. On the Mori cone of blow-ups of the plane. *arXiv:1001.5243.*

[4] M. Dumnicki, A. Küronya, C. Maclean, and T. Szemberg. Rationality of Seshadri constants and the Segre-Harbourne-Gimigliano-Hirschowitz conjecture. *Adv. Math.* 303:1162–1170, 2016.