On intrinsic negative curves

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Definition. Let $\Delta \subseteq \mathbb{R}^n$ be a *lattice polytope*, that is the convex hull of finitely many lattice points of $\mathbb{R}^n$. The projective toric variety X_Δ is the closure of the image of the map

$$x \mapsto [x^p \,:\, p \in \Delta \cap \mathbb{Z}^n],$$

where $x = (x_1, \ldots, x_n)$. Observe that $\dim X_\Delta = \dim \Delta$.

Definition. Let $f \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$ be a irreducible Laurent polynomial whose partial derivatives of order $\leq m-1$ vanish at $(1,1)$ and let Δ be its Newton polygon.

- One says that f defines an *intrinsic negative curve* if

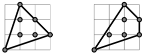
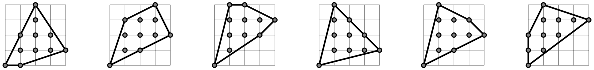
$$2\mathrm{Area}(\Delta) - m^2 < 0.$$

  This curve is the strict transform of the closure of $V(f) \subseteq (\mathbb{C}^*)^2$ in the blowing-up $\tilde{X}_\Delta$ of X_Δ at $(1,1)$.

- The intrinsic negative curve is *expected* if

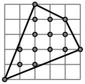
$$|\Delta \cap \mathbb{Z}^2| > \binom{m+1}{2}.$$

There are only a finite number of intrinsic negative curves of bounded volume because the number of non-equivalent polygons is bounded. All the non-equivalent polygons for intrinsic negative curves of multiplicity ≤ 4 are the following.

m	Δ
1	
2	
3	
4	

The above intrinsic curves are expected. In each case $|\partial\Delta \cap \mathbb{Z}^2| = m+1$ so that by Pick's theorem $|\Delta \cap \mathbb{Z}^2| = \binom{m+1}{2} + 1$.

Example. The smallest value of m for an unexpected intrinsic negative curve is 5. The lattice polygon Δ is the following



One has $|\partial\Delta \cap \mathbb{Z}^2| = m - 1$ and $|\partial\Delta \cap \mathbb{Z}^2| = \binom{m+1}{2}$ which imply that the corresponding curve has arithmetic genus 1. The curve is defined by the Laurent polynomial

$$1 - 8xy + 3xy^2 + 6x^2y^4 - x^2y^5 + 3x^2y + 20x^2y^2$$
$$-18x^2y^3 - 18x^3y^2 + 8x^3y^3 + 6x^4y^2 - x^4y^4 - x^5y^2$$

which is the unique one whose Newton polygon is contained in Δ and has multiplicity 5 at $(1, 1)$. Its strict transform in $\tilde{X}_\Delta$ is smooth of genus 1 and self-intersection -1.

Observation. If C is a negative curve with Newton polygon Δ, and $\Delta \subseteq \Delta'$ with $2\text{Area}(\Delta') - m^2 < 0$, then C is negative also in $\tilde{X}_{\Delta'}$.

Theorem ([2]). The surface $X(9, 10, 13)$ does not contain expected negative curves.

Idea of proof. Let Δ be the triangle defined by the ample generator of $\mathbb{P}(a,b,c)$ and let $\Delta_n := n\Delta$.

- The curve $C \sim dH - mE$ is negative if $d/m < \sqrt{abc}$ and it is expected if $|\Delta_d \cap \mathbb{Z}^2| > \binom{m+1}{2}$.
- Let $d = abc\,q + r$ with $0 \leq r < abc$. The above inequalities and the fact that $p(n) := |\Delta_n \cap \mathbb{Z}^2|$ is a Ehrhart quasi-polynomial imply the following inequality:

$$q < \frac{2|\Delta_r \cap \mathbb{Z}^2|}{\sqrt{abc} - (a+b+c)}.$$

- Since q and r are bounded a computer search in the case $(a,b,c) = (9,10,13)$ allows one to conclude that there are no expected negative curves.

$\square$

Definition. Let $a, b \geq 0$, $k \geq 3$ be integers satisfying the equation

$$(a+b)^2 = kab + 1. \tag{1}$$

To each such triple associate an integral triangle $\mathrm{IT}(a,b)$ and a rational triangle $\mathrm{RT}(a,b)$ with vertices:

$$\begin{aligned} \mathrm{IT}(a,b): &\qquad (0,0), (a+b, kb), (a,0), \\ \mathrm{RT}(a,b): &\qquad (0,0), (a, a+b), \left(a - \tfrac{a+b}{k}, 0\right). \end{aligned}$$

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Two families of expected intrinsic curves

Theorem ([1]).

1. Each of these triangles supports a negative curve of multiplicity m at $(1, 1)$, with $m = a + b$ in the integral case and $m = a$ in the rational case.
2. The negative curves corresponding to $IT(a, b)$ for $a \geq b > 0$ and $RT(a, b)$ for $a > b > 1$ are pairwise non-isomorphic.

Idea of proof of 1.

- Each such triangle Δ defines a Laurent polynomial $f(x, y)$ of multiplicity $\geq m$ at $(1, 1)$ because the expected dimension is non-negative. One has to show that $f(x, y)$ is irreducible.
- Each such polygon Δ has a side Δ_0 which is a lattice segment of length one. One shows that Δ_0 is contained in the Newton polygon Δ_f of $f(x, y)$. Then $f(x, y)$ is irreducible because its Newton polygon cannot be sum of two non-trivial polygons.
- To prove the inclusion $\Delta_0 \subseteq \Delta_f$ one makes use of the following lemma.

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Idea of proof of 1.

Lemma. Let S be a set of $\binom{m+1}{2}$ lattice points on the plane. Then, S supports a Laurent polynomial vanishing to order m at $e = (1, 1)$ if and only if there is a degree $m-1$ curve interpolating all points in S.

Idea of proof. When the number of monomials is the same as the number of conditions given by (logaritmic) derivatives, then a nontrivial solution $f(x, y)$ exists if and only if some logarithmic partial derivative p vanishes on all monomials in S when evaluated at e. Now

$$p(x\partial_x, y\partial_y)(x^a y^b)|_{(x,y)=(1,1)} = p(a, b).$$

This p is a polynomial of degree at most $m - 1$ that vanishes at all lattice points in S. $\square$

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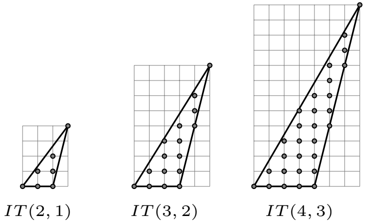
Idea of proof of 1.

- The lattice points of Δ are distributed along vertical segments of cardinality $1, 1, 2, 3 \dots m$, where the two single points correspond to the two vertices of Δ_0. In the following picture we display three such polygons with $k = 4$.



If Δ_f would not contain one of the two vertices of Δ_0, then its set of lattice points would satisfy the hypotheses of the previous lemma. By repeated use of Bezout's theorem there cannot be a plane curve of degree $m-1$ passing through all the lattice points of Δ minus a vertex of Δ_0. $\square$

Idea of proof of 2.

- Let $\xi^{int}_{a,b}$ and $\xi^{rat}_{a,b}$ be the irreducible polynomials defined by $IT(a,b)$ and $RT(a,b)$, respectively and let $\varepsilon^{int}_{a,b}$ and $\varepsilon^{rat}_{a,b}$ be their leading constants. Let $\tau\colon \mathbb{Z}^2 \to \mathbb{Z}^2$ be the linear map $(a,b) \mapsto (b,(k-1)b-(a+b))$. The above polynomials satisfy the following relations

$$\xi^{int}_{a,b} = \xi^{rat}_{a,b}\xi^{rat}_{\tau(a,b)} - \varepsilon^{rat}_{a,b}x^a(y-1)^{a+b},$$
$$\left(\xi^{rat}_{\tau(a,b)}\right)^k = \xi^{int}_{a,b}\xi^{int}_{\tau(a,b)} - \varepsilon^{int}_{a,b}x^{a+b}(y-1)^{kb}.$$

The first equality holds when $a>0$ and the second when $b>0$.

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Idea of proof of 2.

- Let $\Delta := RT(a,b)$. In the toric variety $\tilde{X}_\Delta$ the polynomial $\xi^{rat}_{a,b}$ defines a negative curve C and there is a divisor D with $C \cdot D = 0$ such that

$$\xi^{int}_{a,b}, x^a(1-y)^{a+b} \in H^0(\tilde{X}_\Delta, D).$$

- Thus the two polynomials must be constant multiples of each other modulo $\xi^{rat}_{a,b}$. Write

$$\xi^{int}_{a,b} = \xi^{rat}_{a,b} g - \varepsilon^{rat}_{a,b} x^a (y-1)^{a+b}$$

  for some g with constant term 1, supported in $RT(\tau(a,b))$ and vanishing to order at least b at e. There is only one such polynomial, $g = \xi^{rat}_{\tau(a,b)}$. The second relation is proved in a similar way.

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Idea of proof of 2.

- The Newton polygon of $\xi^{int}_{a,b}$ is $IT(a,b)$. We already showed that the polygon must contain Δ_0, so it remains to prove that

  $$x^a \text{ is a monomial of } \xi^{int}_{a,b}.$$

  This is a direct consequence of the fact that the Minkowski sum $RT(a,b) + RT(\tau(a,b))$ does not contain $(a,0)$, so that the statement follows from

  $$\xi^{int}_{a,b} = \xi^{rat}_{a,b}\xi^{rat}_{\tau(a,b)} - \varepsilon^{rat}_{a,b}x^a(y-1)^{a+b}.$$

- Similar arguments allows one to describe the Newton polygon of $\xi^{rat}_{a,b}$ and thus the statement follows from this description of the Newton polygons. $\square$

Theorem. Conversely, if a negative expected curve of multiplicity $m > 0$ is supported in a triangle Δ such that Δ has

- two integral vertices $(0, 0)$ and (m, h) where m and h are relatively prime,
- a possibly non-integral vertex (r, s) with $0 < r < m$,

then the negative curve is isomorphic to one defined in the previous theorem.

References

[1] Javier González-Anaya, José Luis González, and Kalle Karu. Curves generating extremal rays in blowups of weighted projective planes. *arXiv:2002.07123*.

[2] Kazuhiko Kurano and Naoyuki Matsuoka. On finite generation of symbolic Rees rings of space monomial curves and existence of negative curves. *J. Algebra* 322 (9):3268–3290, 2009.