Classification of wormhole singularities, I

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Motivation

- Let $0 < q < m$ be coprime integers and let $(Q \in Y)$ be a (c.q.s) cyclic quotient singularity $\frac{1}{m}(1, q)$. Kollár and Shepherd-Barron introduced the notion of P-resolutions to study the deformation space of cyclic quotient singularities.

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- Wormhole singularities were introduced in 2021 by Urzúa and Vilches to study this particular wormhole phenomenon in the Kollar–Shepherd-Barron–Alexeev (KSBA) compactification of the moduli space of surfaces of general type. A wormhole singularity is a cyclic quotient singularity which admits at least two different extremal P-resolutions.

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- The classification of these singularities has been open.

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My main contribution is a definition! This definition is the notion of **coherent graph of a framed triangulated polygon**. It provides us with a framework where the problem of classification can be solved, and an alternative proof of the HTU-theorem on the maximum number of extremal P-resolutions that is much more straightforward.

Cyclic quotient singularities

Let $m > 1$ be an integer, and ξ a m-th primitive root of 1. Let $a, b > 0$ be integers coprime to m, and consider the action of $\mathbb{Z}/m\mathbb{Z}$ on $\mathbb{C}^2$ given by

$$T\colon \mathbb{C}^2 \to \mathbb{C}^2, (x, y) \mapsto (\xi^a x, \xi^b y)$$

Definition 1.

A cyclic quotient singularity ($c.q.s$) is a singularity isomorphic to the germ at 0 of the quotient $\mathbb{C}^2/\langle T\rangle$ for some integers m, a and b. We will denote the singularity as $\frac{1}{m}(a, b)$.

It is enough to restrict our attention to the case $\frac{1}{m}(1, q)$. Its resolution is constructed with toric methods and it is summarized as follows:

Theorem 2.

Let $p \in X$ be a c.q.s $\frac{1}{m}(1, q)$, where X is a neighborhood of p and $\phi\colon Y \to X$ its minimal resolution. Then $Exc(\phi)$ is a collection of rational curves $E_1, \cdots, E_r$ such that $E_i \cdot E_{i+1} = 1$ for $i \in \{1, \cdots, r-1\}$, $E_i^2 = -e_i$ and $E_i \cdot E_j = 0$ for any other case, where

$$\frac{m}{q} = [e_1, \cdots, e_r] = e_1 - \cfrac{1}{e_2 - \cfrac{1}{\ddots \, \frac{1}{e_r}}}$$

is its Hirzebruch-Jung continued fraction (each e_i is an integer greater than 1).

T-singularities

Definition 3.

A T-singularity is a quotient singularity which admits a $\mathbb{Q}$-Gorenstein one parameter smoothing.

Proposition 1.

T-singularities are either ADE or c.q.s $\frac{1}{dn^2}(1, dna-1)$ *with* $0 < a < n$, $d \geq 1$ *and* $gcd(n, a) = 1$. *When* $d = 1$, *we call them Wahl singularities.*

Definition 4.

A T-chain is the exceptional divisor of the minimal resolution of a T-singularity (non-ADE). For a Wahl singularity, we call it a W-chain.

A result of Wahl gives us an algorithm to obtain all (non-ADE) T-singularities. Furthermore, it tells us how a T-singularity can be recognized from its minimal resolution.

T-singularities

Proposition 2 (Algorithm for T-chains).

For any (non-ADE) T-singularity $\frac{1}{dn^2}(1, dna-1)$*, we have:*

(i) If $n=2$*, then* $[4]$ *and* $[3,2,\cdots,2,3]$ *where the number of* $2's$ *in this expression is* $d-2$*, are T-chains.*

(ii) If $[b_1,\cdots,b_r]$ *is a T-chain, then so are* $[2,b_1,\cdots,b_{r-1},b_r+1]$ *and* $[b_1+1,b_2,\cdots,b_r,2]$*.*

(iii) Any T-chain can be obtained by starting with one of the T-chains in (i) *and iterating the steps described in* (ii)*.*

The T-chain associated with $\frac{1}{dn^2}(1, dna-1)$ will be denoted as $[d\binom{n}{a}]$. The operations in (ii) will be called $\mathcal{L}$-operations and $\mathcal{R}$-operations, respectively. The proof of this result also shows that

$$\left[d\binom{2n-a}{n}\right] \xleftarrow{\mathcal{L}} \left[d\binom{n}{a}\right] \qquad \text{y} \qquad \left[d\binom{n}{a}\right] \xrightarrow{\mathcal{R}} \left[d\binom{n+a}{a}\right]$$

T-singularities

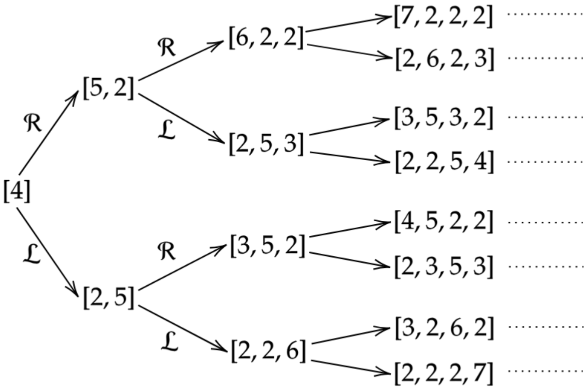

Figure: W-chains.

P-resolutions

Definition 5.

Let $P \in Y$ be a c.q.s. A P-resolution of $P \in Y$ is a **partial resolution** $f \colon X \to Y$ over P such that X only have T-singularities and K_X is ample relative to f.

P-resolutions

Definition 5.

Let $P \in Y$ be a c.q.s. A P-resolution of $P \in Y$ is a **partial resolution** $f\colon X \to Y$ over P such that X only have T-singularities and K_X is ample relative to f.

Example 6 (Non-example of a P-resolution).

Consider the cyclic quotient singularity $\frac{1}{13}(1,3)$ on X, and $\phi\colon Y \to X$ its minimal resolution. In the chain corresponding to the H-J continued fraction $\frac{13}{3} = [5,2,2]$ we can do 3 blow-ups to get the following chain of $\mathbb{P}^1$'s

$$[6,2,2,1,5,2]$$

on a surface Z. After contracting these W-chains onto a surface W, it defines a partial resolution $f\colon W \to X$ that is not a P-resolution of the singularity $\frac{1}{13}(1,3)$ (K_X is not ample relative to f).

P-resolutions

Definition 7.

Let $\frac{m}{q} = [e_1, \cdots, e_r]$ be a H-J continued fraction. Its dual fraction is

$$\frac{m}{m-q} := [b_1, \cdots, b_s].$$

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From the Riemenschneider's dot diagram, if we write

$$\frac{m}{q} = [\underbrace{2, \cdots, 2}_{a_1}, b_1, \underbrace{2, \cdots, 2}_{a_2}, b_2, \cdots, \underbrace{2, \cdots, 2}_{a_{c-1}}, b_{c-1}, \underbrace{2, \cdots, 2}_{a_c}]$$

where $a_i \geq 0$ and $b_i \geq 3$ for all i, then

$$\frac{m}{m-q} = [a_1 + 2, \underbrace{2, \cdots, 2}_{b_1 - 3}, a_2 + 3, \underbrace{2, \cdots, 2}_{b_2 - 3}, a_3 + 3, \cdots, a_{c-1} + 3, \underbrace{2, \cdots, 2}_{b_{c-1} - 3}, a_c + 2]$$

P-resolutions

Each fraction together with its dual induce a zero continued fraction. Namely, if $\frac{m}{q} = [e_1, \cdots, e_r]$ and $\frac{m}{m-q} = [b_1, \cdots, b_s]$ then:

$$[b_s, \cdots, b_1, 1, e_1, \cdots, e_r] = 0$$

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Key relation 1: There is a well-known one-to-one correspondence between zero continued fractions and triangulated polygons with a hidden index.

Theorem 8.

Let $(b_1, \cdots, b_s)$ be a sequence of positive integers. Then $[b_1, \cdots, b_s] = 0$ if and only if there exists b_0 such that $(b_0, b_1, \cdots, b_s)$ is the vector of indices of some triangulated $(s+1)$-gon.

The number b_0 is called the hidden index of the triangulation associated with the zero continued fraction $[b_1, \cdots, b_s] = 0$.

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Given a triangulated $(s+1)$-gon with vertices $P_0, \cdots, P_s$, one defines the index of a vertex P_i as

$$v_i := (\text{number of diagonals from the vertex } P_i) + 1.$$

Definition 9.

We say that a triangulated polygon $\mathscr{P}$ is a **framed triangulated polygon** if we specify a hidden index.

P-resolutions

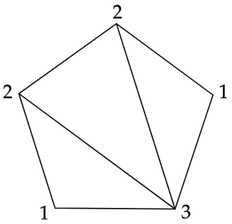

Figure: Triangulated pentagon.

We have the 0−fractions: $[3,1,2,2]$, $[1,2,2,1]$, $[2,2,1,3]$,$[2,1,3,1]$, and $[1,3,1,2]$.

P-resolutions

Lemma 10.

Let $\mathcal{P}$ be a convex polygon with $s+1$ sides. The indices from the vertices of $\mathscr{P}$ will be taken mod $s+1$. Consider $[b_1, \cdots, b_s] = 0$ for a triangulation of $\mathcal{P}$, then

1. $b_0 + b_1 \cdots + b_s = 3(s-1)$.
2. *At least two b_i must be equal to 1. Furthermore, for $s \geq 3$, the entries equal to 1 cannot be in consecutive positions.*

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2. *At least two b_i must be equal to 1. Furthermore, for $s \geq 3$, the entries equal to 1 cannot be in consecutive positions.*

By the relation (1), the number b_0 is completely determined by the linear relation

$$b_0 := 3(s-1) - \sum_{i=1}^{s} b_i.$$

Therefore, we can write the zero continued fraction $[b_1, \cdots, b_s]$ as $[b_1, \cdots, b_s \mid b_0]$. The latter notation is the extended zero chain.

Extremal P-resolutions

Definition 11.

Let $0 < q < m$ be coprime integers, and let $(Q \in Y)$ be a cyclic quotient singularity $\frac{1}{m}(1, q)$. An extremal P-resolution of $(Q \in Y)$ is a partial resolution $f\colon (C \subset X) \to (Q \in Y)$, such that X has only Wahl singularities, there is one exceptional curve C and isomorphic to $\mathbb{P}^1$, and K_X is ample relative to f.

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Theorem 12 (Hacking-Tevelev-Urzúa, 2013).

Let $0 < q < m$ be coprime integers. If $\frac{m}{m-q} = [k_1, \cdots, k_s]$ where $k_i \geq 2$, then there is a bijection between extremal P-resolutions associated with the cyclic quotient singularity $\frac{1}{m}(1,q)$ and pairs $1 \leq \alpha < \beta \leq s$ such that

$$[k_1, \cdots, k_{\alpha-1}, k_\alpha - 1, k_{\alpha+1}, \cdots, k_{\beta-1}, k_\beta - 1, k_{\beta+1}, \cdots, k_s] = 0.$$

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$$[k_1, \cdots, k_{\alpha-1}, k_\alpha - 1, k_{\alpha+1}, \cdots, k_{\beta-1}, k_\beta - 1, k_{\beta+1}, \cdots, k_s] = 0.$$

Definition 13.

A sequence $\{b_1, \cdots, b_s\}$, $b_i > 1$ is a WW-sequence if there exists $1 \leq \alpha < \beta \leq s$ such that

$$[b_1, \cdots, b_\alpha - 1, \cdots, b_\beta - 1, \cdots, b_s] = 0.$$

The numbers α and β are called the indices of the WW-sequence, and we say that the zero continued fraction $0 = [b_1, \cdots, b_\alpha - 1, \cdots, b_\beta - 1, \cdots, b_s]$ is a WW-decomposition of the Hirzebruch-Jung continued fraction $[b_1, \cdots, b_s]$.

Wormholes

Definition 14.

A wormhole singularity is a cyclic quotient singularity $\frac{1}{m}(1,q)$ with $0 < q < m$ coprime integers, which admits at least two distinct extremal P-resolutions. Equivalently, the Hirzebruch-Jung continued fraction of $\frac{m}{m-q}$ has at least two different WW-decompositions.

Examples of wormhole singularities

Example 15.

$$\frac{m}{q} := \frac{900}{241} = [4,4,5,2,2,3,2,2].$$

Then

$$\frac{m}{m-q} = [2,2,3,2,3,2,2,5,4].$$

We have the extremal P-resolutions

$$[\binom{11}{3}] - (1) - [\binom{19}{5}] \quad \text{associated with} \quad [2,2,3,2,2,2,1,5,4] = 0.$$

$$[\binom{26}{7}] - (1) - [\binom{4}{1}] \quad \text{associated with} \quad [2,2,3,2,3,1,2,4,4] = 0.$$

It is obvious that the hidden index is $b_0 = 1$ for this pair.

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Lemma 16.

Let $\{k_1, \cdots, k_s\}$ be a WW-sequence. Assume that $0 = [b_1, \cdots, b_s]$ and $0 = [b'_1, \cdots, b'_s]$ are WW-decompositions of $[k_1, \cdots, k_s]$. If $[b_1, \cdots, b_s \mid v_0]$ and $[b'_1, \cdots, b'_s \mid v'_0]$ are the corresponding extended zero chains, then $v_0 = v'_0$. This number v_0 is called the WW-index of the WW-sequence $\{k_1, \cdots, k_s\}$.

Basic wormhole singularities

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Example 18.

$$\frac{m}{q} := \frac{800}{289} = [3, 5, 2, 2, 7, 2, 2, 2].$$

Then

$$\frac{m}{m-q} := [2, 3, 2, 2, 5, 2, 2, 2, 2, 5].$$

We have the extremal P-resolutions:

$$\bullet - (3) - [\binom{17}{4}] \quad \text{associated with} \quad [1, 3, 2, 2, 5, 1, 2, 2, 2, 5] = 0.$$

$$[\binom{5}{2}] - (2) - [\binom{5}{1}] \quad \text{associated with} \quad [2, 3, 1, 2, 5, 2, 2, 2, 1, 5] = 0.$$

Computing the hidden index b_0, we get that $b_0 = 2$ for this pair.

$$0 = [1, 3, 2, 2, 5, 1, 2, 2, 2, 5 \mid 2]$$

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1. Reduce the classification of wormhole singularities to the classification of basic wormhole singularities (HTU-algorithm).
2. Define basic wormhole triangulations, and notice that we can classify basic wormhole singularities by classifying basic wormhole triangulations.
3. Candidates to basic wormhole triangulations: framed accordion triangulations.
4. Introduce the coherent graph of a framed triangulated polygon and its properties.
5. Define a standard frame of an accordion triangulation. Then define standard family of accordion triangulations and coherent rotation of diagonals.
6. Classify framed accordion triangulations with a standard frame that are basic wormhole triangulations.
7. Justify how we can change the frame from the classification in the step above to classify all basic wormhole triangulations.

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If $\mathscr{P}$ is a basic wormhole triangulation associated with a basic wormhole singularity, then any other triangulation associated with other WW-decomposition of the same basic wormhole singularity is called a companion of $\mathscr{P}$.

Main results

The main result is that the wormhole condition is equivalent to the consistency of an explicit system of linear relations naturally associated with accordion triangulations.

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Theorem 20 (N., 2025).

Let $n \geq 2$ be a integer, and let $\mathscr{P}$ be a framed accordion triangulation with a standard frame and weights $x_1, \cdots, x_n$. Then $\mathscr{P}$ is a basic wormhole triangulation with a standard frame if and only if there exists an integer $1 \leq m \leq n-1$ such that the system $S_0 \cup S_m$ of $2n$ linear equations

$$\begin{aligned} S_0 &: y_i = x_{n-i (mod\ n)} - k_i^{(0)} && \text{for } 1 \leq i \leq n, \\ S_m &: y_i = x_{(n-i)+m\ (mod\ n)} - k_i^{(m)} && \text{for } 1 \leq i \leq n, \end{aligned}$$

is consistent; where the numbers $k_i^{(0)}$ and $k_i^{(m)}$ are equal to 3 for all i, except at four specific positions determined by n and m, where they are equal to 1.

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is consistent; where the numbers $k_i^{(0)}$ and $k_i^{(m)}$ are equal to 3 for all i, except at four specific positions determined by n and m, where they are equal to 1.

Explicitly, the numbers $k_i^{(0)}$ and $k_i^{(m)}$ are given as follows:

- **If n is odd**: $k^{(0)}_{\frac{n-1}{2}} = 1,\ k^{(0)}_n = 1,\ k^{(m)}_{\frac{n-1}{2}+\lceil \frac{m}{2} \rceil \text{ (mod n)}} = 1,\ k^{(m)}_{n+\lfloor \frac{m}{2} \rfloor \text{ (mod n)}} = 1$, otherwise $k_i^{(0)} = 3$ and $k_i^{(m)} = 3$.
- **If n is even**: $k^{(0)}_{\frac{n}{2}} = 1,\ k^{(0)}_n = 1,\ k^{(m)}_{\frac{n}{2}+\lfloor \frac{m}{2} \rfloor \text{ (mod n)}} = 1, k^{(m)}_{n+\lfloor \frac{m}{2} \rfloor \text{ (mod n)}} = 1$, otherwise $k_i^{(0)} = 3$ and $k_i^{(m)} = 3$.

Main results

The consistency of this system of linear relations depends on explicit numerical conditions involving n and m. If $S_0 \cup S_m$ is consistent, we give a parametric solution depending on $\gcd(n, m)$-parameters and explain how to recover a parametric family of basic wormhole triangulations with a standard frame from this parametric solution.

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Corollary 21 (N. ,2025).

We give a 3-step constructive algorithm to obtain all basic wormhole triangulations with a fixed number of weights.
Input*: An integer* $n \geq 2$*.*
Output*: All parametric families of basic wormhole triangulations with* n *weights.*

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Input: *An integer $n \geq 2$.*
Output: *All parametric families of basic wormhole triangulations with n weights.*

As an application of these ideas we give a simple alternative proof of the Hacking-Tevelev-Urzúa theorem on the maximal number of extremal P-resolutions.

Theorem 22.

A cyclic quotient singularity $\frac{1}{m}(1, q)$ can admit at most two distinct extremal P-resolutions.

Accordion triangulation

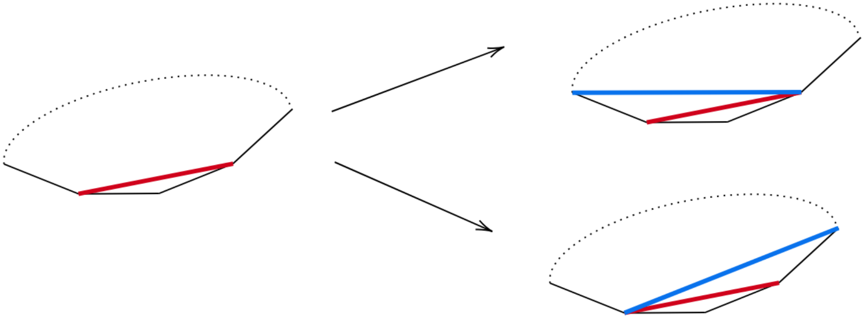

Figure: Construction of an accordion triangulation.

Accordion triangulation

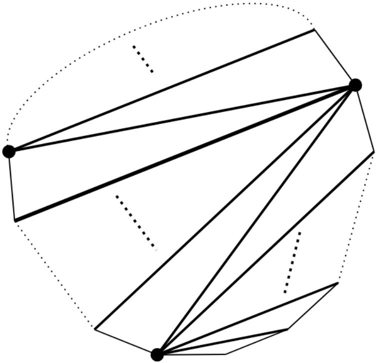

Figure: Construction of an accordion triangulation.

Accordion triangulation

Definition 23.

An accordion triangulation of $s+1$ sides with base at b_0 is a triangulated $(s+1)$-gon $\mathscr{P}$ with vertices $b_0, b_1, \cdots, b_s$ constructed recursively as follows:

- **Step I**: Let $\mathcal{P}$ be a convex polygon with vertices $b_0, b_1, \cdots, b_s$. Draw the diagonal ℓ_0 from b_0 to b_2, and define Δ_0 as the triangle with vertices b_0, b_1 and b_2. Set $J = 1$ and continue to Step II.
- **Step II**: We have two options:
  - If $1 \leq J \leq (s+1) - 4$: Let $\mathscr{P}^{(J)}$ be the polygon obtained from removing $\bigcup_{k=0}^{J-1} (\Delta_k \setminus \ell_k)$ from $\mathcal{P}$. Let b'_0 and b'_1 be the vertices in $\ell_{J-1} \cap \mathcal{P}$. Let b'_L be the vertex adjacent to b'_0 in $\mathscr{P}^{(J)}$ that is not b'_1, and let b'_R be the vertex adjacent to b'_1 in $\mathscr{P}^{(J)}$ that is not b'_0. Choose the diagonal ℓ_J as either the diagonal from b'_0 to b'_R or the diagonal from b'_1 to b'_L. Define Δ_J as the triangle in $\mathscr{P}^{(J)}$ determined by ℓ_J, redefine J as $J+1$, and return to Step II.
  - If $J = (s+1) - 3$: Continue to Step III.
- **Step III**: Let $\mathscr{P}$ be the convex polygon $\mathcal{P}$ triangulated with the diagonals $\ell_0, \ell_1, \cdots, \ell_{(s+1)-4}$.

An accordion triangulation is an accordion triangulation of $s+1$ sides with base at b_0 for some $s \geq 3$ and some vertices $b_0, b_1, \cdots, b_s$.

Accordion triangulation

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An accordion triangulation is an accordion triangulation of $s+1$ sides with base at b_0 for some $s \geq 3$ and some vertices $b_0, b_1, \cdots, b_s$.

Accordion triangulations do not come with a frame !

Accordion triangulation

Lemma 24.

Let $\mathscr{P}$ *be a triangulated polygon. Then* $\mathscr{P}$ *has exactly two index* 1 *vertices if and only if* $\mathscr{P}$ *is an accordion triangulation.*

Accordion triangulation

Lemma 24.

Let $\mathscr{P}$ be a triangulated polygon. Then $\mathscr{P}$ has exactly two index 1 vertices if and only if $\mathscr{P}$ is an accordion triangulation.

Proposition 3.

Let $\mathscr{P}$ be a basic wormhole triangulation. Then $\mathscr{P}$ is a framed accordion triangulation.

Coherent graph. Story...

I think this idea is a product of sadness and luck.

Coherent graph. Story...

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Figure: UGA main library bus stop.

Coherent graph

Now we introduce the key definition of coherent graph of a framed triangulated polygon.

Definition 25.

Let $\mathscr{P} = [b_1, \cdots, b_s \mid b_0]$ be a framed triangulated $(s+1)$-gon. Reading the vector $v = (b_1, \cdots, b_s, b_0)$ from left to right, define x_i as the i-th entry that is greater than 2, and j_i as the position such that $b_{j_i} = x_i$. Let n be the number of $x_i's$ associated with the vector v. For each $1 \leq i \leq n-1$, define y_i as $j_{i+1} - j_i - 1$. Define y_n as the number $(s+1) - n - \sum_{i=1}^{n-1} y_i$. Construct the coherent graph $G_{\mathscr{P}}$ as follows:

- **Vertices**: $G_{\mathscr{P}}$ has n vertices, one for each x_i. The vertex associated with x_i is denoted by x_i.
- **Edges**: For $1 \leq i \leq n$, there is an edge between x_i and x_{i+1}, labeled with y_i. Additionally, there is an edge between x_n and x_1, labeled with y_n.

The vertices of $\mathscr{P}$ corresponding to the $x_i's$ are called the weights of $\mathscr{P}$. We denote the coherent graph of $\mathscr{P}$ by $G_{\mathscr{P}} = ([x_1, \cdots, x_{n-1} \mid x_n], (y_1, \cdots, y_n))$. If only the weights of $\mathscr{P}$ are relevant, we use the simplified notation $G_{\mathscr{P}} = [x_1, \cdots, x_{n-1} \mid x_n]$.

Coherent graph

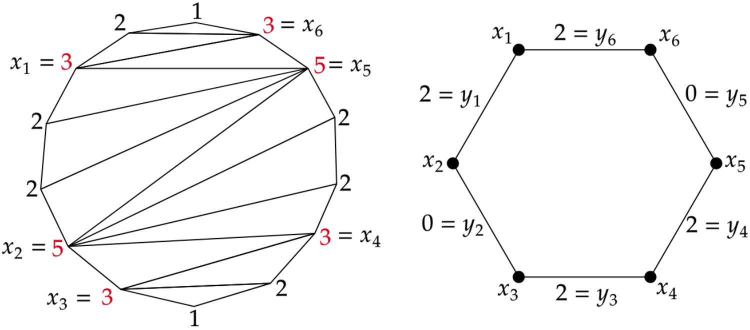

Figure: An example of a framed accordion triangulation $\mathscr{P}$ and its coherent graph $G_{\mathscr{P}}$.

Note that $\mathscr{P} = [1, 2, 3, 2, 2, 5, 3, 1, 2, 3, 2, 2, 5 \mid 3]$,
$\mathscr{P}_1 = [2, 3, 2, 2, 5, 3, 1, 2, 3, 2, 2, 5, 3 \mid 1]$, and $\mathscr{P}_2 = [3, 2, 2, 5, 3, 1, 2, 3, 2, 2, 5, 3, 1 \mid 2]$
have the same coherent graph.

Coherent graph of framed accordion triangulations

The coherent graph is defined for any framed triangulated $(s+1)$-gon, however, the most powerful applications of these graphs are for framed accordion triangulations. In this case, the coherent graph is naturally endowed with a very explicit set of linear relations.

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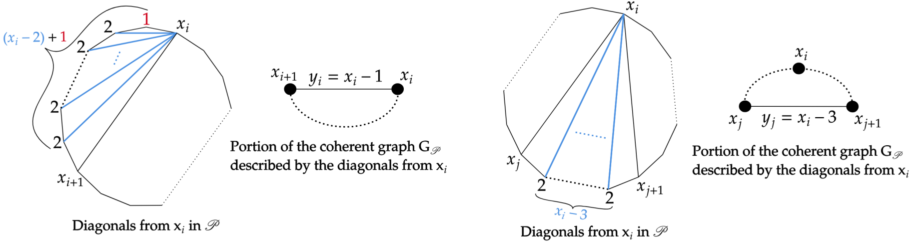

Diagonals from x_i in $\mathscr{P}$

Portion of the coherent graph $G_{\mathscr{P}}$ described by the diagonals from x_i

Diagonals from x_i in $\mathscr{P}$

Portion of the coherent graph $G_{\mathscr{P}}$ described by the diagonals from x_i

Figure: Geometry of accordion triangulation.

Coherent graph of framed accordion triangulations

Definition 26.

We say that diagonals from the weight x_i go to the pair $(x_{j \pmod n}, x_{j+1 (\bmod n)})$ if all the diagonals in $\mathscr{P}$ from x_i fit between x_j and x_{j+1}.

Coherent graph of framed accordion triangulations

Definition 26.

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Lemma 27 (Alternative description of accordion triangulations).

Let $n \geq 2$ be an integer, let $x_1, \cdots, x_n$ be integers greater than 2, and let $\mathscr{P}_{x_1,\cdots,x_n}$ be the family of all accordion triangulations with weights $x_1, \cdots, x_n$. For each $1 \leq i \leq n$ and $1 \leq j \leq n$, there exists a unique accordion triangulation in $\mathscr{P}_{x_1,\cdots,x_n}$ whose diagonals from x_i go to the pair $(x_{j \pmod n}, x_{j+1 \pmod n})$.

Coherent graph of framed accordion triangulations

Lemma 28.

Let $\mathscr{P}$ *be a framed accordion triangulation and let* $G_{\mathscr{P}} = ([x_1, \cdots, x_{n-1} \mid x_n], (y_1, \cdots, y_n))$ *be its coherent graph. If the diagonals from the weight* x_i *go to the pair* $(x_{j \pmod n}, x_{j+1 \pmod n})$ *for some pair of indices* $1 \leq i, j \leq n$, *then there is a linear system of* n *relations associated with the graph* $G_{\mathscr{P}}$. *Specifically,*

$$y_\ell = x_{\ell_{i,j}} - k_{\ell_{i,j}} \quad \text{for} \quad 1 \leq \ell \leq n, \tag{1}$$

where each weight in $\mathscr{P}$ *is exactly one of the numbers* $x_{\ell_{i,j}}$ *for some appropriate* ℓ *, and* $k_{\ell_{i,j}}$ *is equal to* 3 *for all* ℓ, *except for two special values of* ℓ *for which* $k_{\ell_{i,j}}$ *is equal to* 1.

Coherent graph of framed accordion triangulations

Lemma 28.

Let $\mathscr{P}$ be a framed accordion triangulation and let $G_{\mathscr{P}} = ([x_1, \cdots, x_{n-1} \mid x_n], (y_1, \cdots, y_n))$ be its coherent graph. If the diagonals from the weight x_i go to the pair $(x_{j \pmod n}, x_{j+1 \pmod n})$ for some pair of indices $1 \leq i, j \leq n$, then there is a linear system of n relations associated with the graph $G_{\mathscr{P}}$. Specifically,

$$y_\ell = x_{\ell_{i,j}} - k_{\ell_{i,j}} \quad \text{for} \quad 1 \leq \ell \leq n, \tag{1}$$

where each weight in $\mathscr{P}$ is exactly one of the numbers $x_{\ell_{i,j}}$ for some appropriate ℓ, and $k_{\ell_{i,j}}$ is equal to 3 for all ℓ, except for two special values of ℓ for which $k_{\ell_{i,j}}$ is equal to 1.

By the alternative description of accordion triangulations the system of linear relations above does not depend on the choice of the weight x_i, it depends only on $\mathscr{P}$.

Coherent graph of framed accordion triangulations

Definition 29.

Let $\mathscr{P}$ and $\widetilde{\mathscr{P}}$ be framed triangulated polygons. We say that the coherent graphs

$$G_{\mathscr{P}} = ([x_1, \cdots, x_{n-1} \mid x_n], (y_1, \cdots, y_n)), \qquad G_{\widetilde{\mathscr{P}}} = ([\widetilde{x}_1, \cdots, \widetilde{x}_{m-1} \mid \widetilde{x}_m], (\widetilde{y}_1, \cdots, \widetilde{y}_m))$$

are **equal** if and only if $n = m$, $x_i = \widetilde{x}_i$ and $y_i = \widetilde{y}_i$ for $1 \leq i \leq n$.

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are **equal** if and only if $n = m$, $x_i = \widetilde{x}_i$ and $y_i = \widetilde{y}_i$ for $1 \leq i \leq n$.

Lemma 30.

Let $\mathscr{P}$ and $\widetilde{\mathscr{P}}$ be basic wormhole triangulations that are not equal. Assume that the hidden index in both triangulations is a weight. The coherent graphs $G_{\mathscr{P}}$ and $G_{\widetilde{\mathscr{P}}}$ are equal if and only if $\widetilde{\mathscr{P}}$ is a companion of $\mathscr{P}$.

Coherent graph of framed accordion triangulations

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are **equal** if and only if $n = m$, $x_i = \widetilde{x}_i$ and $y_i = \widetilde{y}_i$ for $1 \leq i \leq n$.

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Let $\mathscr{P}$ and $\widetilde{\mathscr{P}}$ be basic wormhole triangulations that are not equal. Assume that the hidden index in both triangulations is a weight. The coherent graphs $G_{\mathscr{P}}$ and $G_{\widetilde{\mathscr{P}}}$ are equal if and only if $\widetilde{\mathscr{P}}$ is a companion of $\mathscr{P}$.

This is the key lemma. It tells us that the condition of being a basic wormhole triangulation is equivalent to the consistency of two system of linear equations!

Thank you!