NEGATIVE CONTINUED FRACTIONS IN BIRATIONAL GEOMETRY: A GUIDE TO DEGENERATIONS OF SURFACES WITH WAHL SINGULARITIES

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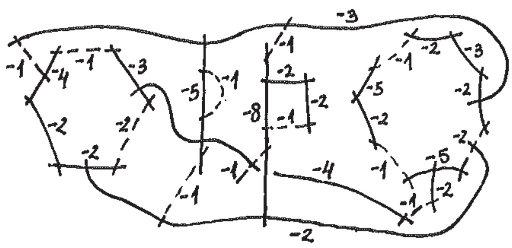

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Introduction

Our aim is to provide a guide in a single text about the theoretical framework regarding specific degenerations of complex surfaces. We will be explicit about computations, which are intricately governed by Hirzebruch-Jung continued fractions

$$e_1 - \cfrac{1}{e_2 - \cfrac{1}{\ddots - \cfrac{1}{e_r}}}.$$

The exploration of these degenerations traces back to Jonathan Wahl [W1, W2], who pioneered the study of singularities arising in the degenerated surfaces discussed in this text (see also [LW]). Wahl [W1] focused on analyzing 2-dimensional normal singularities that admit smoothings where the dual of the dualizing sheaf lifts. Cyclic quotient singularities $\frac{1}{n^2}(1, na-1)$ with $\gcd(n, a) = 1$ are examples [W1, (2.7) Theorem], we call them Wahl singularities. In the subsequent article [W2], Wahl studied smoothings using their Milnor numbers, and showed [W2, Examples (5.9.1)] that the above smoothings of Wahl singularities had Milnor number 0, hence the Milnor fiber had the rational homology of a disk. A relevant property is that these smoothings preserve the self-intersection of the canonical class [LW]; more generally, the cyclic quotient singularities of type $\frac{1}{dn^2}(1, dna-1)$, where $\gcd(n, a) = 1$ [LW, Proposition 5.9], are the only ones which are quotients of smoothings of Gorenstein singularities. These properties are key in the deformations employed by Kollár and Shepherd-Barron [KSB] to compactify the moduli space of surfaces of general type. These authors called the ADE singularities and the above cyclic quotients T-singularities [KSB, Definition 3.7], and proved that they were the only 2-dimensional quotients admitting such "$\mathbb{Q}$-Gorenstein smoothings". Among them, the most important for deformations are Wahl singularities. These singularities were also considered by Kawamata [K3, Section 10] in the context of surface degenerations (see also [K2]). Important applications were obtained in the thesis work of Manetti (see for example [M1, M2]). Concurrently, Fintushel and Stern [FS], and Park [P2] developed the rational blow-down construction. This construction is the diffeomorphic analogue of degenerations involving Wahl singularities [SSW], and had a great impact on the construction of exotic 4-manifolds. Hacking's thesis work [H1] develops more on this theory of KSB deformations, and applies this to the moduli of plane curves [H2]. Then we have the paper [LP1] by Lee and Park with the construction of simply-connected Campedelli surfaces via singular rational surfaces with only T-singularities. This opened the door to many applications and further development of the underlying theory of $\mathbb{Q}$-Gorenstein smoothings. To exemplify, we have [H5], [H4], [HP], [PPS1], [PPS2], [LN], [PSU1], [U4], [U3], [SU1], [RTU], [PPSU], [ES], [RU1], [UV], [RU2], [DRU], [RU3], [TU], [EU], [FRU], [UZ1], [UZ2]. Behind the KSB moduli space, we have Mori theory [KM1]. Continuing the study of semistable extremal neighborhoods by Mori, Kollár, and Prokhorov [M4], [KM2], [M3], [MP] (see also Kawamata [K3]) we have [HTU] which gives a way to explicitly run the minimal model program to degenerations of surfaces with only log terminal singularities. Part of the consequences mentioned above use this as a main tool. Pending is an explicit birational theory for nonnormal degenerations with orbifold normal crossing singularities [H3].

Degenerations with only Wahl singularities are related to various open problems, for example:

(1) *Markov's uniqueness conjecture* (see [A1] as the main reference, [UZ1] with the connections to Algebraic Geometry together with the Exercises §1.2 (7), (8), (9)). Markov numbers are final outputs of the birational geometry of these degenerations, just as the projective plane is for the classical birational geometry of algebraic surfaces.

(2) *Horikawa's famous problem* (see [E] [MNU]). Horikawa [H9] provided a complete classification of nonsingular projective surfaces with $K^2 = 2p_g - 4$. We also know the topological and diffeomorphism type, except for each of the families with $K^2 = 16t$. In this case, there are two connected components for the moduli space that parametrize homeomorphic surfaces, but it is unknown if they are diffeomorphic (see the end of the introduction in [H9]). One strategy is to construct a common degeneration with only Wahl singularities as in [M2], which would prove that they are diffeomorphic. There is only one degeneration in the literature [LP2] for one of the families. The problem was recently studied in [MNU] from that point of view, showing that it cannot be addressed through T-degenerations. A similar Horikawa problem is stated in [CP, Section 8] for surfaces with $K^2 = 2p_g - 3$ and $p_g = 4t + 1$.

(3) *Exotic blow-ups of $\mathbb{CP}^2$ at few points* (see for example [RU3] and Exercise §2.3(6)). Given a closed smooth 4-manifold N, a closed smooth 4-manifold is exotic if it is homeomorphic but not diffeomorphic to N. This phenomenon does not appear in dimensions ≤ 3, it is controlled in dimensions ≥ 5, but is wild in dimension 4. Famous examples are the spheres of dimension n and the corresponding work of Kervaire-Milnor. There are exotic blow-ups of $\mathbb{CP}^2$ at n points for any $n \geq 2$. For $n = 1$ is an open question. For $n = 8, 7, 6, 5, 4$ they can be constructed from singular surfaces, where $n = 4$ has very few examples [RU3]. It depends on the existence of very special configurations of rational curves.

(4) *Kollár conjecture* (see [K2], [dJ] and Remark 3.10). Kollár–Shepherd-Barron [KSB] classified deformations of quotient singularities by means of P-resolutions. This is a result that involves birational geometry. It is believed that a similar statement should work for any rational singularity, although the new P-resolutions may not only have T-singularities, and may not be normal. See [PS1, JS] for sandwiched singularities.

(5) *Optimal bounds for T-singularities for rational surfaces* (see [RU1, FRU] and Exercise §2.3 (7)). By Alexeev's boundedness [A2], there is a finite list of T-singularities for all surfaces W with log-canonical singularities, big and nef canonical class, and K_W^2 smaller than a fixed constant. There are optimal bounds in [RU1, FRU] when the surface is not rational. The rational case is open. See the bound in [RU1] for a rational surface, which depends on the degree of configurations of rational curves.

(6) *Existence of simply-connected $p_g = 0$ surfaces of general type with $K^2 \geq 5$* (see [LP1, RU3]). In [LP1, PPS1, PPS2] and other papers, the key to have complex smoothings is not to have local-to-global obstructions to deform. When $p_g = 0$, this implies $K^2 \leq 4$. Therefore, we need to deal with obstructions if $K^2 \geq 5$. In [RU3] there are candidates for $K^2 = 5$.

(7) *Wahl conjecture* (see [W3], [W4]). The claim is that a 2-dimensional normal surface singularity that admits a smoothing with Milnor number equal to 0 must be in the list of weighted homogeneous singularities [BS] (see also [SSW]). See the attempt in the pre-print [PSS], and the recent pre-print [B2].

(8) *Classification of Uroburos* (see §1.3 and Exercises (10) and (11)). This is related to the wormhole conjecture [UV].

(9) *Problems on semi-orthogonal decompositions of derived categories of* $p_g = 0$ *surfaces* (see [TU] and §6). As an example, check the beautiful [LT, Conjecture 1.9].

(10) *Coble-Mukai lattice for* $p_g \neq 0$ (see [U2]; Theorem 4.4). Although it works for any $p_g = 0$ surface, the challenge is to describe it and find geometric applications in the case of $p_g \neq 0$.

In the exercises: $\star$ means challenge, $\star\star$ means open question.

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1. HIRZEBRUCH-JUNG CONTINUED FRACTIONS

The negative continued fractions in this section appeared in the work of Jung [J] on resolutions of two-dimensional cyclic quotient singularities. This was rediscovered by Hirzebruch in [H8]. Negative continued fractions are, as we will see, closely related to classical Dedekind sums. A nice account of this is [M6], with references as old as the 1895 paper [V1]. A beautiful book on Dedekind sums and geometry is [HZ].

1.1. Basics.

Definition 1.1. Let $\{e_1, \ldots, e_r\}$ be positive integers. We say that it admits a *Hirzebruch-Jung continued fraction* (HJ continued fraction)

$$[e_1, \ldots, e_r] := e_1 - \cfrac{1}{e_2 - \cfrac{1}{\ddots - \frac{1}{e_r}}},$$

if $[e_i, \ldots, e_r] > 0$ for all $i \geq 2$, and $[e_1, \ldots, e_r] \geq 0$. Its *value* is the rational number $[e_1, \ldots, e_r]$, and its *length* is r.

For example, $\{2, 1, 1, 2\}$ does not admit an HJ continued fraction since $[1, 1, 2] < 0$. Neither $\{1, 1, 2\}$, but $\{1, 2\}$ does. The sequence $\{2, 1, 3, 2\}$ admits and its value is $[2, 1, 3, 2] = \frac{1}{3}$. The positive requirement on partial fractions allows us to operate with *blow-downs* and *blow-ups*. These operations are defined and proved to be well defined in the following lemma.

Lemma 1.2. *Given an HJ continued fraction $[\ldots, u, 1, v, \ldots]$ that is not $[1, 1]$ or $[1]$, we have the blow-down HJ continued fraction $[\ldots, u-1, v-1, \ldots]$, and conversely, given an HJ continued fraction $[\ldots, u, v, \ldots]$, the blow-up $[\ldots, u+1, 1, v+1, \ldots]$ is an HJ continued fraction. (This includes the cases $[1, v, \ldots]$ and $[\ldots, u, 1]$.) Moreover, we have equality on values*

$$[\ldots, u, 1, v, \ldots] = [\ldots, u-1, v-1, \ldots]$$

when the blow-down 1 is not in the first position.

Proof. Let $[e_1, \ldots, e_r]$ be an HJ continued fraction with $e_i = 1$. Say $i = 1$. Then, by definition, $e_2 > 1$ and $[e_2, \ldots, e_r] \geq 1$. Then $[e_2 - 1, e_3, \ldots, e_r]$ is an HJ continued fraction. The cases $i > 1$ will follow from $[\ldots, u, 1, v, \ldots] = [\ldots, u-1, v-1, \ldots]$, and this last property is a consequence of the identity

$$u - \frac{1}{v} = u + 1 - \frac{1}{1 - \frac{1}{v+1}}.$$

□

A sequence $\{e_1, \ldots, e_r\}$ with $e_i \geq 2$ for all i gives an HJ continued fraction whose value is a rational number $[e_1, \ldots, e_r] > 1$. For example $[7] = 7$, $[3, 2, 4] = \frac{17}{7}$, $[2, \ldots, 2] = \frac{m+1}{m}$ where m is the number of 2s. In fact, this gives a one-to-one correspondence between $[e_1, \ldots, e_r]$ with $e_i \geq 2$ and rational numbers greater than 1. In this way, for any coprime integers $0 < q < m$ we can associate a unique HJ continued fraction

$$\frac{m}{q} = [e_1, \ldots, e_r]$$

with $e_i \geq 2$ for all i.

Definition 1.3. An HJ continued fraction is said to be *minimal* if it is equal to $[1,1]$, $[1]$, or $[e_1,\ldots,e_r]$ with $e_i \geq 2$ for all i.

Lemma 1.4. *Any HJ continued fraction can be reduced via blow-downs into a unique minimal HJ continued fraction.*

Proof. Exercise §1.1(2). □

We note that for minimal HJ continued fractions, the possible values correspond to $[1,1]=0$, $[1]=1$, and $\mathbb{Q}_{>1}$.

Remark 1.5. As an algebraic geometer reader might have guessed already, this is directly related with the Castelnuovo theorem in two-dimensional birational geometry: $[1]=1$ represents a (-1)-curve (or a nonsingular point after contracting), $[1,1]=0$ represents a fiber in a $\mathbb{P}^1$-fibration, and $\mathbb{Q}_{>1}$ represents the so-called two-dimensional cyclic quotient singularities. All of this will be central in the next sections.

Definition 1.6. Let $0<q<m$ be coprime integers and $\frac{m}{q}=[x_1,\ldots,x_r]$ with $x_i \geq 2$ for all i. Its *dual* is

$$\frac{m}{m-q}=[y_1,\ldots,y_s]$$

with $y_i \geq 2$ for all i.

Proposition 1.7. *Let $0<q<m$ be coprime integers. Consider $\frac{m}{q}=[x_1,\ldots,x_r]$ and its dual $\frac{m}{m-q}=[y_1,\ldots,y_s]$. Then $[x_1,\ldots,x_r,1,y_s,\ldots,y_1]=0$.*

Proof. Exercise §1.1 (3). □

Proposition 1.8. *Let $0<q<m$ be coprime integers, and consider $\frac{m}{q}=[e_1,\ldots,e_r]$. Then,*

$$\begin{bmatrix} m & -q^{-1} \\ q & \frac{1-qq^{-1}}{m} \end{bmatrix} = \begin{bmatrix} e_1 & -1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} e_r & -1 \\ 1 & 0 \end{bmatrix},$$

where $0<q^{-1}<m$ is the integer that satisfies $qq^{-1}\equiv 1(\text{mod } m)$.

Proof. Exercise §1.1 (4). □

Remark 1.9. Let $0<q<m$ be coprime integers. The (classical) *Dedekind sum*[1] associated to (q,m) is

$$s(q,m) := \sum_{i=1}^{m-1} \left(\left(\frac{i}{m}\right)\right)\left(\left(\frac{iq}{m}\right)\right)$$

where $((x)) = x-[x]-\frac{1}{2}$ for any rational number x. (The symbol $[x]$ is the integral part of x.) There is a well-known relation (see for example [M6]) with HJ continued fractions. If $\frac{m}{q}=[e_1,\ldots,e_r]$, then

$$12s(q,m)=\sum_{i=1}^{r}(e_i-3)+\frac{q+q^{-1}}{m}.$$

This can also be proved using the Noether's formula $12\chi(\mathcal{O}_S)=K_S^2+\chi_{\text{top}}(S)$ on a suitable algebraic surface S [U1, Section 3]. Various identities between Dedekind sums

[1]Richard Dedekind (1831–1916) was the mathematician who introduced Dedekind sums to express the functional equation of the Dedekind eta function.

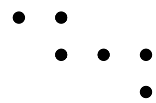
can be deduced via geometry. For example, the *reciprocity law of Rademacher* is implied by the rationality of weighted projective planes $\mathbb{P}(a, b, c)$, or we have the formula [U5]

$$s(q, m) = s(q + 1, m) + s(q^{-1} + 1, m) + \frac{m-1}{m},$$

which is also a consequence of rationality of a surface. The values of these Dedekind sums have been studied by various authors, among them Girstmair [G1, G2]. He proved a peculiar large-scale behavior that has been used in [U5, U1]. For a particular use of Dedekind sums and computations on surfaces, see [RU4], [U6], [UY].

Exercises.

(1) Show that HJ continued fractions $[e_1, \ldots, e_r]$ with $e_i \geq 2$ for all i are in one-to-one correspondence with $\mathbb{Q}_{>1}$ through their values.
(2) Prove that any HJ continued fraction can be reduced using blow-downs to a unique minimal HJ continued fraction (Lemma 1.4).
(3) Let $0 < q < m$ be coprime integers. Consider $\frac{m}{q} = [x_1, \ldots, x_r]$ and its dual $\frac{m}{m-q} = [y_1, \ldots, y_s]$. Prove $[x_1, \ldots, x_r, 1, y_s, \ldots, y_1] = 0$ (Proposition 1.7). (Hint: Use blow-downs.) This defines the *dots diagrams of Riemenschneider* [R2]: For $\frac{m}{q} = [x_1, \ldots, x_r]$ we draw rows of $x_i - 1$ points, such that we draw the first point of a row below the last point of the previous row. Then by adding the dots in the columns we obtain $y_i - 1$ for $\frac{m}{m-q} = [y_1, \ldots, y_s]$. For example, the dots diagram for $\frac{19}{7} = [3, 4, 2]$ is



and so $\frac{19}{12} = [2, 3, 2, 3]$. Therefore, we always have $\sum_{i=1}^{r} x_i - \sum_{i=1}^{s} y_i = r - s$. In fact, if we write

$$\frac{m}{q} = [\underbrace{2, \ldots, 2}_{a_1}, b_1, \underbrace{2, \ldots, 2}_{a_2}, b_2, \ldots, \underbrace{2, \ldots, 2}_{a_{e-1}}, b_{e-1}, \underbrace{2, \ldots, 2}_{a_e}],$$

where $a_i \geq 0$ and $b_i \geq 3$ for all i, then

$$\frac{m}{m-q} = [a_1 + 2, \underbrace{2, \ldots, 2}_{b_1 - 3}, a_2 + 3, \underbrace{2, \ldots, 2}_{b_2 - 3}, a_3 + 3, \ldots, a_{e-1} + 3, \underbrace{2, \ldots, 2}_{b_{e-1} - 3}, a_e + 2].$$

(4) Let $0 < q < m$ be coprime integers, and consider $\frac{m}{q} = [e_1, \ldots, e_r]$. Show that

$$\begin{bmatrix} m & -q^{-1} \\ q & \frac{1 - qq^{-1}}{m} \end{bmatrix} = \begin{bmatrix} e_1 & -1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} e_r & -1 \\ 1 & 0 \end{bmatrix},$$

where $0 < q^{-1} < m$ is the inverse of q mod m (Proposition 1.8).
(5) Let $0 < q < m$ be coprime integers, and consider $\frac{m}{q} = [e_1, \ldots, e_r]$. Show that $\frac{m}{q^{-1}} = [e_r, \ldots, e_1]$.
(6) Let $0 < q < m$ be coprime integers. Consider $\frac{m}{q} = [x_1, \ldots, x_r]$ and its dual $\frac{m}{m-q} = [y_1, \ldots, y_s]$. Show that $\frac{m^2}{mq-1} = [x_1, \ldots, x_r + y_s, \ldots, y_1]$, and

$$\frac{m^2}{m(m-q)+1} = [y_1, \ldots, y_s, 2, x_r, \ldots, x_1].$$

(7) Let $0 < q < m$ be coprime integers, and consider $\frac{m}{q} = [e_1, \ldots, e_r]$. Define the matrix

$$M_i := \begin{bmatrix} -e_i & 1 & & & \\ 1 & -e_{i+1} & 1 & & \\ & 1 & -e_{i+2} & & \\ & & & \ddots & 1 \\ & & & 1 & -e_r \end{bmatrix}.$$

Show that $m = (-1)^r \det(M_1)$ and $q = (-1)^{r-1}\det(M_2)$. In fact, we have $-\frac{\det(M_i)}{\det(M_{i+1})} = [e_i, \ldots, e_r]$ for all $i = 1, \ldots, r-1$.

1.2. **Wahl chains.**

We now focus on special HJ continued fractions: Wahl chains. The word "chain" puts an emphasis on the sequence of numbers whose negatives will be the self-intersections of a chain of $\mathbb{P}^1$s. This will be seen when we study resolution of singularities.

Definition 1.10. Let d, n, a be positive integers that satisfy $d \geq 1$, $n \geq 1$, and $0 < a \leq n$ with $\gcd(n, a) = 1$. A *T-chain* is the sequence $\{e_1, \ldots, e_r\}$ in $\frac{dn^2}{dna-1} = [e_1, \ldots, e_r]$. A *Wahl chain* is a T-chain with $d = 1$. For $d > 1$, an A_{d-1} *chain* is a T-chain with $n = 1$.

Remark 1.11. T-chains are the numerical data for the minimal resolution of T-singularities, introduced by Kollár–Shepherd-Barron [KSB, Definition 3.7] (we are ignoring Du Val singularities of type D and E). For example, T-singularities are used in [KSB] to understand all deformations of two-dimensional quotient singularities. As we shall see, T-singularities are naturally dominated by Wahl singularities.

T-chains are well understood. When $n = 1$ they are [1], [2], [2, 2], ..., $[2, \ldots, 2] = \frac{d}{d-1}$, where we have $d - 1$ 2s. For $n > 1$, we have the following algorithmic description for all T-chains [KSB, Proposition 3.11], originally due to Wahl.

Proposition 1.12. *The T-chains with $n > 1$ are*

(i) either $[4]$ ($d = 1$), or $\frac{4d}{2d-1} = [3, 2, \ldots, 2, 3]$ where 2 appears $d - 2$ times ($d > 1$),
(ii) or, it is obtained by starting with one of the singularities in (i) and iterating the operations $[2, e_1, \ldots, e_{r-1}, e_r + 1]$ or $[e_1 + 1, e_2, \ldots, e_r, 2]$ many times.

Proof. Exercise §1.2(1). □

In Figure 1, we represent this algorithm for Wahl chains. If $\frac{dn^2}{dna-1} = [e_1, \ldots, e_r]$, then $\frac{dn^2}{dn(n-a)-1} = [e_r, \ldots, e_1]$, and so we think of them as the same T-chain.

Corollary 1.13. *If $\frac{dn^2}{dna-1} = [e_1, \ldots, e_r]$ and $n > 1$, then $r - d + 2 = \sum_{i=1}^{r}(e_i - 2)$.*

Proof. Exercise §1.2(3). □

We will later talk about Markov's uniqueness conjecture, whose "geometry" is very much related to our purposes (see [UZ1]). A *Markov triple* is a positive integer solution (a, b, c) of the Markov equation

$$x^2 + y^2 + z^2 = 3xyz.$$

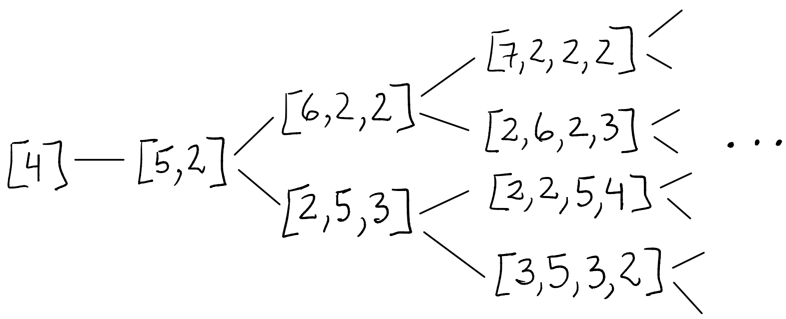


FIGURE 1. The Wahl tree.

The coordinates are called *Markov numbers*. These solutions appear in various places in mathematics, see the book [A1]. Note that permutations of coordinates in a Markov triple is a Markov triple. Also the *mutation*

$$(a, b, c) \mapsto (a, b, 3ab - c)$$

sends Markov triples into Markov triples. It turns out that symmetries and mutations generate all Markov triples starting from $(1, 1, 1)$, and define the Markov tree in Figure 2. The 111 years old and famous *Markov conjecture* (known also as the Frobenius Uniqueness Conjecture [F1]) states that in a Markov triple (a, b, c) where $a, b < c$ the integer c determines the integers a, b. Markov conjecture has been checked for Markov numbers up to 10^{15000} [P4]. In the exercises you can read about 3 equivalences.

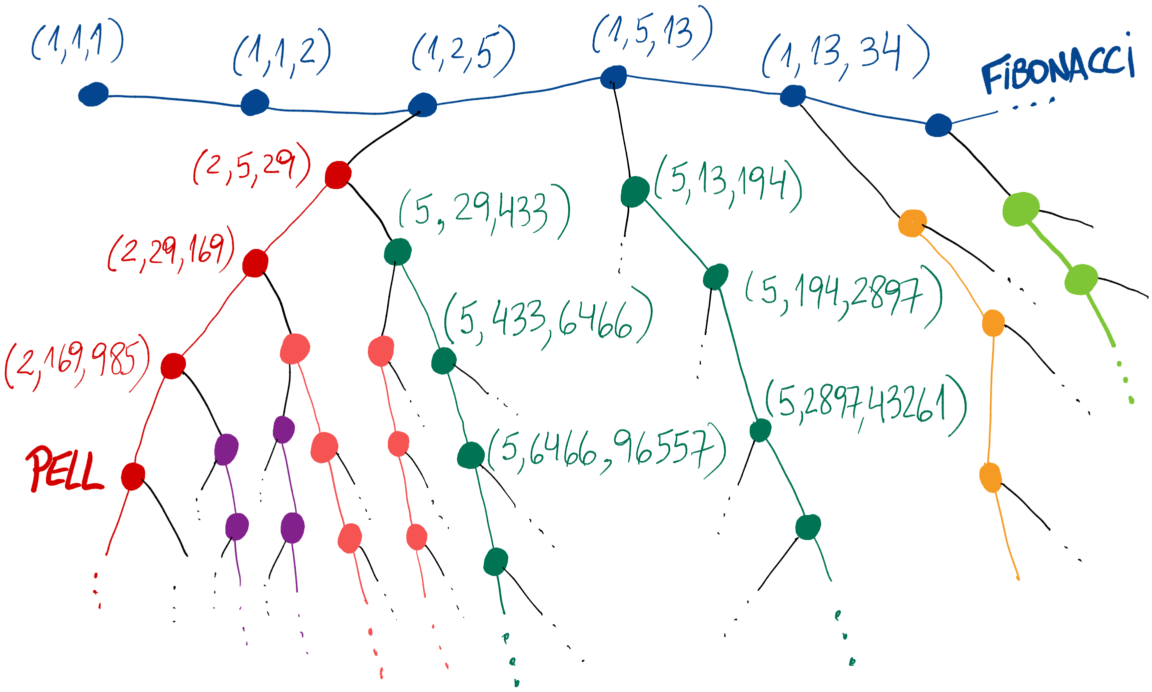


FIGURE 2. The Markov tree.

Exercises.

(1) Prove Proposition 1.12.
(2) Show that $\frac{dn^2}{dna-1} = [e_1, \ldots, e_r]$ implies $\frac{dn^2}{dn(n-a)-1} = [e_r, \ldots, e_1]$.
(3) Prove that if $\frac{dn^2}{dna-1} = [e_1, \ldots, e_r]$ and $n > 1$, then $r - d + 2 = \sum_{i=1}^{r}(e_i - 2)$ (Corollary 1.13).
(4) Show that the ns of the Wahl chains $[4]$, $[5,2]$, $[2,5,3]$, $[3,5,3,2]$, $[2,3,5,3,3]$, ... are precisely the Fibonacci numbers.
(5) Consider a T-chain for some (d,n,a) of length r. Show that

$$n \leq F_{r-d}$$

where F_i is the ith Fibonacci number defined by the recursion $F_{-2} = 1$, $F_{-1} = 1$, and

$$F_i = F_{i-1} + F_{i-2}$$

for $i \geq 0$. Find a characterization for equality.
(6) Define Wahl-2 chains by the same algorithm as in (ii) Proposition 1.12 but starting with $[2]$. Examples: $[3,2]$, $[2,4,2,3]$, $[3,2,2,4,2,5,2]$. Show that an HJ continued fraction $\frac{m}{q} = [e_1, \ldots, e_r]$ is a Wahl-2 chain if and only if $q^2 \equiv -1 (\mathrm{mod} m)$. In this way, for Wahl-2 chains, we have $[e_1, \ldots, e_r, 1, e_1, \ldots, e_r] = 0$. These Wahl-2 chains are relevant for weights of Markov numbers [UZ1, §3].
(7) $\star\star$ (Markov's uniqueness conjecture) Given an integer m, there are at most two $0 < q < m$ coprime such that

$$\frac{m}{q} = \left[\frac{m_0}{q_0}, 4, \frac{m_1}{q_1}\right]$$

where $\frac{m}{q}$, $\frac{m_0}{q_0}$ and $\frac{m_1}{q_1}$ are the fractions of Wahl-2 chains. [2]
(8) $\star\star$ (Markov's uniqueness conjecture) Given an integer m, there are at most two $0 < q < m$ coprime such that

$$\frac{m^2}{mq-1} = [\mathtt{W}_0^\vee, 10, \mathtt{W}_1^\vee]$$

for some Wahl chains $\mathtt{W}_i$, where $\mathtt{W}_i^\vee$ is the corresponding Wahl dual.
(9) $\star\star$ (Markov's uniqueness conjecture) Given an integer m, then there are at most two $0 < q < m$ coprime such that

$$\left[5, \frac{m}{q}, 2, \frac{m}{m-q^{-1}}, 5\right] = [\mathtt{W}_0, 2, \mathtt{W}_1]$$

for some Wahl chains $\mathtt{W}_0$ and $\mathtt{W}_1$, where $0 < q^{-1} < m$ and $qq^{-1} \equiv 1 \pmod{m}$.
(10) Show that (a,b,c) is a Markov triple if and only if

$$s(a^{-1}b, c) = s(b^{-1}c, a) = s(c^{-1}a, b) = 0$$

where $s(x,y)$ is the classical Dedekind sum for (x,y) (see Remark 1.9), and for $(x^{-1}y, z)$ we mean x^{-1} inverse of x mod z. See more in [HZ, p.160].

[2]There is an abuse of notation: $[\frac{m}{q}, \ldots]$ means $[e_1, \ldots, e_r, \ldots]$ where $\frac{m}{q} = [e_1, \ldots, e_r]$.

1.3. **Zero continued fractions.**

HJ continued fractions with value equal to 0 will be called *zero continued fractions*. They are key to work with deformations of cyclic quotient singularities and with the birational geometry of the degenerations that we are going to study in these notes.

By Lemma 1.4, we find all zero continued fractions through blowing-up starting with $[1,1]$, and so the list begins:

$[1,1]$,
$[1,2,1]$, $[2,1,2]$,
$[1,2,2,1]$, $[2,1,3,1]$, $[1,3,1,2]$, $[3,1,2,2]$, $[2,2,1,3]$,
etc...

There is a well-known one-to-one correspondence between zero continued fractions and triangulations of polygons [C2, S9, HTU]. A *triangulation of a convex polygon* $P_0\, P_1 \cdots P_r$ is given by drawing some non intersecting diagonals on it which divide the polygon into triangles. For a fixed triangulation, v_i is defined as the number of triangles that have P_i as one of its vertices. Note that

$$v_0 + v_1 + \ldots + v_r = 3(r-1).$$

Using an easy induction on r, one can show that $[k_1, \ldots, k_r]$ is a zero continued fraction if and only if there exists a triangulation of $P_0\, P_1 \cdots P_r$ such that $v_i = k_i$ for every $1 \leq i \leq r$. In this way, the number of zero continued fractions of length r is the *Catalan number*

$$\frac{1}{r}\binom{2(r-1)}{r-1}.$$

r	2	3	4	5	6	7	8	9	10
Number of zero continued fractions of lenght r	1	2	5	14	42	132	429	1430	4862

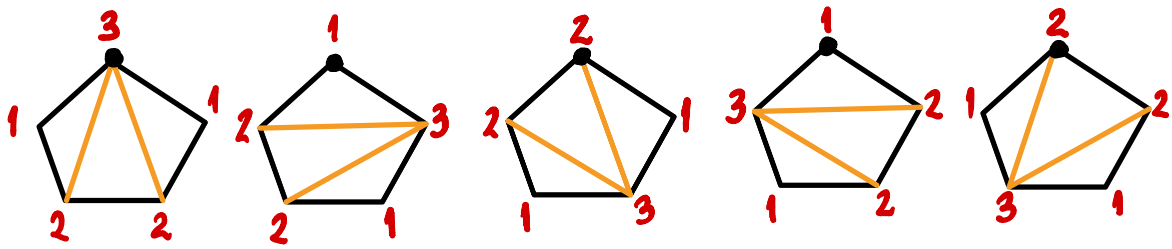


FIGURE 3. All the triangulations of a pentagon and the v_i's.

We will see that to know about the components of the deformation space of a cyclic quotient singularity, we need to know about the zero continued fractions that its dual HJ continue fraction admits.

Definition 1.14. We say that $[e_1, \ldots, e_r]$ with $e_i \geq 2$ admits a *zero continued fraction* of weight λ if there are indices $i_1 < i_2 < \ldots < i_u$ for some $u \geq 1$ and integers $d_{i_k} \geq 1$ such that

$$[\ldots, e_{i_1} - d_{i_1}, \ldots, e_{i_2} - d_{i_2}, \ldots, e_{i_u} - d_{i_u}, \ldots] = 0,$$

and $\lambda + 1 = \sum_{k=1}^{u} d_{i_k}$.

Example 1.15. We have that $\frac{19}{12} = [2,3,2,3]$ admits exactly 3 zero continued fractions:

- $d_1 = 1, d_2 = 1, d_4 = 2$ and so $[2-1, 3-1, 2, 3-2] = [1,2,2,1] = 0$.
- $d_1 = 1, d_3 = 1, d_4 = 1$ and so $[2-1, 3, 2-1, 3-1] = [1,3,1,2] = 0$.
- $d_2 = 1, d_3 = 1$ and so $[2, 3-1, 2-1, 3] = [2,2,1,3] = 0$.

Example 1.16. Each $\frac{m}{q} = [e_1, \ldots, e_r]$ admits at least one zero continued fraction of weight $\lambda = \sum_{i=1}^{r}(e_i - 2) - 1$. This is because we can subtract to obtain $[1, 2, \ldots, 2, 1]$. Sometimes, this is the only one. For example, for every fraction $\frac{m}{m-1}$, except for $\frac{4}{3} = [2,2,2]$ which admits two: $[1,2,1]$ and $[2,1,2]$.

Remark 1.17. The set of admissible zero continued fractions of $[e_1, \ldots, e_r]$ is a subset of all zero continued fractions of length r. Hence, if $e_i \geq r-1$ for all i, then the number of admissible zero continued fractions is the corresponding Catalan number $\frac{1}{r}\binom{2(r-1)}{r-1}$.

Proposition 1.18. *The HJ continued fraction $[e_1, \ldots, e_r]$ admits a zero continued fraction of weight 0 if and only if $[e_1, \ldots, e_r]$ is the dual of a T-chain.*

Proof. This follows from Proposition 1.7 and Exercise §1.1(6). □

We now analyze HJ continued fractions that admit zero continued fractions of weight 1. It turns out that they are central to the birational geometry of degenerations.

Proposition 1.19. *Let $0 < q < m$ be coprime integers. If $\frac{m}{m-q} = [b_1, \ldots, b_s]$ admits a zero continued fraction of weight 1, then the HJ continued fraction of $\frac{m}{q}$ is a T-chain with $d = 2$ and $n > 1$, or*

$$\frac{m}{q} = [\mathtt{W}_0, c, \mathtt{W}_1]$$

where $c \geq 1$ and $\mathtt{W}_i$ are Wahl chains (including one or both empty).

Proof. Exercise §1.3(4). □

After we interpret this geometrically, this last proposition can be a sufficient and necessary characterization when the weight is 1 under a positive condition on the canonical class. (This will correspond to extremal P-resolutions.) It would not be true without that positive condition. Also, there may be infinitely many ways to write $\frac{m}{q} = [\mathtt{W}_0, c, \mathtt{W}_1]$, but there is only one that is "positive", in the sense of birational geometry. This will be a flip.

Definition 1.20. Given a $\frac{m}{q} = [e_1, \ldots, e_r]$ that admits a zero continued fraction of weight 1 for indices $i_1 < i_2$ (and so $d_{i_1} = d_{i_2} = 1$), we define $\delta = 1$ if $i_2 = i_1 + 1$, or

$$\frac{\delta}{\epsilon} = [e_{i_1+1}, \ldots, e_{i_2-1}].$$

This is, δ is the numerator of this "intermediate" HJ continued fraction.

In principle δ depends on $i_1 < i_2$ and $\frac{m}{q}$, but soon we will see that it only depends on $\frac{m}{q}$.

As we saw above, a zero continued fraction of length r defines a triangulation of a polygon with $r+1$ sides. Say that $\frac{m}{q}$ admits the zero continued fraction $[\ldots, e_{i_1} - 1, \ldots, e_{i_2} - 1, \ldots] = 0$. For the corresponding triangulation, we have $v_i = e_i$ except for $v_{i_1} = e_{i_1} - 1$, $v_{i_2} = e_{i_2} - 1$, and

$$v_0 = 3r - 3 - \sum_{i=1}^{r} e_i + 2 = \sum_{i=1}^{r}(3 - e_i) - 1 \geq 1.$$

(Hence, we must have $\sum_{i=1}^r e_i \leq 3r-2$.) We have two types:

(non-minimal type) $v_0 = 1$: In this case, we must have $v_1, v_r \geq 2$ unless the polygon is a triangle (i.e. $r=2$). Therefore, if $r>2$, then i_1, i_2 cannot be $1, r$ since one of the v_{i_k} must be equal to 1. Choose the vertex P from P_1, P_r that does not correspond to i_1, i_2. We now remove the triangle P_r, P_0, P_1, and in this new polygon, we choose as new P_0 the vertex P. We have a new triangulation and a new continued fraction of length $r-1$ that admits a zero continued fraction of weight 1. We continue with this algorithm until we reach a triangulation with $v_0 > 1$.
(minimal type) $v_0 > 1$: In this case $v_{i_1} = 1 = v_{i_2}$. They are the only $v_k = 1$.

In this way, any $\frac{m}{q}$ that admits a zero continued fraction of weight 1 can be constructed from a minimal type fraction.

Is it possible that a fraction $\frac{m}{q}$ admits more than one zero continued fraction of weight 1?

Theorem 1.21. *Let $0<q<m$ be coprime integers. Then $\frac{m}{q}$ admits at most two zero continued fractions of weight 1. If it admits two, then the corresponding δs are equal.*

Proof. See Theorems 4.3 and 4.4 in [HTU]. See also [UV, Section 2]. □

Example 1.22. We have that $\frac{16}{7}$ admits two, as $\frac{16}{7} = [3,2,2,3]$ and so the indices $1<3$ and $2<4$ are two pairs. Similarly with $\frac{40}{31} = [2,2,2,4,2,2,2]$ and the pairs $1<5$, $3<7$ work. The situation can also be non symmetric. For example, $\frac{36}{23} = [2,3,2,2,4]$ works for $2<4$ and $3<5$.

Definition 1.23. A fraction $\frac{m}{q}$ that admits two zero continued fractions of weights 1 is called *wormhole fraction*. [3]

Proposition 1.24. *(Crossing property) A wormhole fraction whose pairs of indices are $i_1 < j_1$ and $i_2 < j_2$ satisfies $i_2 < i_1 < j_2 < j_1$ or $i_1 < i_2 < j_1 < j_2$.*

Proof. See [V3, §3.4]. □

In particular, there are no wormholes with $\delta = 1$.

A wormhole fraction $\frac{m}{q}$ can be reduced to a minimal wormhole fraction. First we note that the corresponding v_0 are equal for both zero continued fractions. So, if it is equal to 1, then we reduce it to a wormhole fraction with $v_0 > 1$. Therefore, minimal wormhole fractions are the main blocks to construct any wormhole fraction. They are peculiar; we name them as *Uroboros* [4] since one can express them with pictures that look like Uroboros.

Picture of an uroboro: Consider a wormhole fraction $\frac{m}{q} = [e_1, \ldots, e_r]$ with indices $i_1 < i_2 < j_1 < j_2$ and corresponding $v_0 > 1$. Let $e_0 := v_0$. (Recall that we must have $e_{i_1} = e_{i_2} = e_{j_1} = e_{j_2} = 2$.) We now draw in a circular way the consecutive numbers $e_i > 2$, and for chains of $e_j = 2$ we draw an arc label by the numbers of 2s. If it is just one 2, then we draw 2. Finally, we draw four dots indicating the positions of $e_{i_1} = e_{i_2} = e_{j_1} = e_{j_2} = 2$, and we join by a segment the dots of $i_1 < j_1$ and the dots of $i_2 < j_2$.

[3] The reason for the name can be found in [UV, V3].

[4] The name and pictures were created by Jonny Evans.

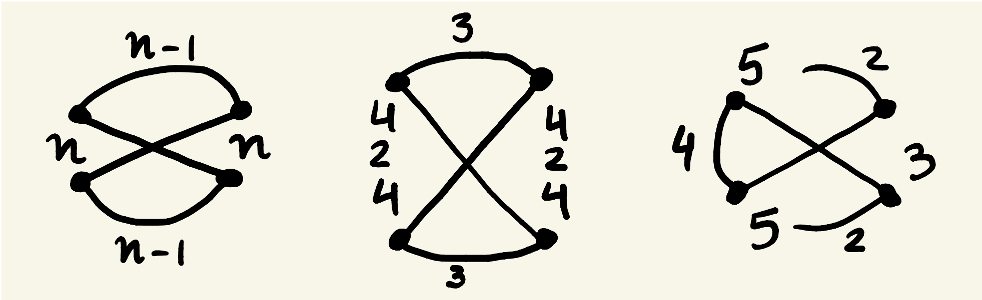


FIGURE 4. Examples of uroboros.

In Figure 4 we have uroboros corresponding to: For $n > 2$,

$$[2, \ldots, 2, n, 2, \ldots, 2]$$

($n-1$ 2s both sides; $v_0 = n$), $[4, 2, 2, 2, 4, 2, 4, 2, 2, 2, 4]$ ($v_0 = 2$), and $[2, 2, 5, 2, 2, 2, 2, 5, 2, 2]$ ($v_0 = 3$). Of course, these uroboros are also uroboros for other HJ continued fractions, just by choosing another v_0 in the circle (not for the distinguished indices). The corresponding δs are $n^2 - 2n + 2$, 58, and 30. There are also less symmetric uroboros, as in Figure 5.

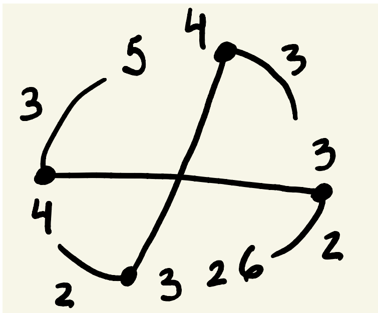


FIGURE 5. Less symmetric uroboro with $\delta = 350$. There are 3 distinct uroboros for $\delta = 350$.

We note that for some δ, there may be more than one uroboro. The first δ with this property is 130, where there are two. For $\delta = 4930$ there are 8 distinct uroboros. Using a computer, one can show that the list of δs less than or equal to 1000 is:

5 10 13 17 26 30 34 37 50 53 58 65 68 82 89
101 122 130 145 170 178 185 197
219 222 226 233 257 290
317 325 327 338 350 362
401 442 457 466 485
520 530 577 578
610 626 677
730 738 785
842 853
901 962 964 986 997.

Exercises.

(1) Confirm that there is a bijection between zero continued fractions and triangulations of polygons.

(2) Let $0 < q < m$ coprime integers, and $\frac{m}{q} = [e_1, \ldots, e_r]$ with $r \geq 2$ and $e_i \geq r-1$. Show that $\frac{m}{q}$ admits $\frac{1}{r}\binom{2(r-1)}{r-1}$ zero continued fractions.

(3) Prove Proposition 1.18.

(4) $\star$ Let $0 < q < m$ be coprime integers. If $\frac{m}{q}$ admits a zero continued fraction of weight 1 and indices $i_1 < i_2$, then

$$\frac{m}{q} = [\mathtt{W}_0, c, \mathtt{W}_1]$$

where $c \geq 1$ and $\mathtt{W}_i$ are Wahl chains (including one or both empty) (Proposition 1.19). Show that if $\frac{n_i^2}{n_i a_i - 1} = \mathtt{W}_i$, then

$$m = n_0^2 + n_1^2 \pm \delta n_0 n_1.$$

(5) $\star$ Assume that $\frac{m}{q}$ admits a zero continued fraction of weight 1 and indices $i_1 < i_2$, and so $\frac{m}{q} = [\mathtt{W}_0, c, \mathtt{W}_1]$, where $\frac{n_i^2}{n_i a_i - 1} = \mathtt{W}_i$ are Wahl chains, and $c \geq 1$. Show that

$$\delta = |(c-1)n_0 n_1 + n_1 a_0 - n_0 a_1|.$$

(6) $\star$ Let $\{e_1, \ldots, e_r\}$ be integers with $e_k \geq 2$. Prove that there are at most two pairs of indices $i < j$ such that

$$[\ldots, e_i - 1, \ldots, e_j - 1, \ldots] = 0.$$

This shows that $\frac{m}{q}$ admits at most two zero continued fractions of weight 1.

(7) Prove that a wormhole fraction whose pairs of indices are $i_1 < j_1$ and $i_2 < j_2$ satisfies either $i_2 < i_1 < j_2 < j_1$ or $i_1 < i_2 < j_1 < j_2$ (Proposition 1.24).

(8) Use Figure 6 to classify all wormhole fractions with $\delta = 2$.

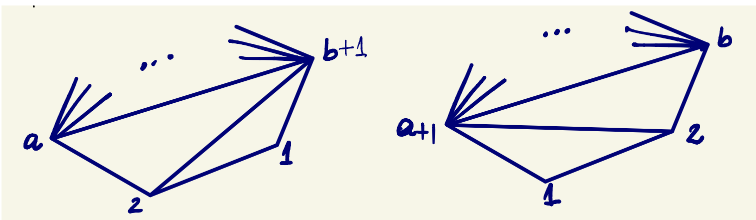


FIGURE 6. Wormholes for $\delta = 2$.

(9) Show that $\delta = 3$ is not possible for wormholes. Same for $\delta = 4, 6, 7, 8, 9$.

(10) $\star\star$ Find a classification for Uroboros.

(11) $\star\star$ Find a classification for the deltas of wormhole fractions. What can be said about the denominator ϵ? (See Definition 1.20.)

This is Exercise §1.2 9 about the Markov conjecture when $m = 29$.
We list q, and m/q, $m/(m - q^{-1})$ between **5 2 5**.

1 **5** [29] **2** [2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2] **5**
2 **5** [15, 2] **2** [3, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2] **5**
3 **5** [10, 3] **2** [2, 3, 2, 2, 2, 2, 2, 2, 2, 2] **5**
4 **5** [8, 2, 2, 2] **2** [5, 2, 2, 2, 2, 2, 2] **5**
5 **5** [6, 5] **2** [2, 2, 2, 3, 2, 2, 2, 2] **5**
6 **5** [5, 6] **2** [2, 2, 2, 2, 3, 2, 2, 2] **5**
7 5 [5, 2, 2, 2, 2, 2, 2] 2 [8, 2, 2, 2] 5
8 **5** [4, 3, 3] **2** [2, 3, 3, 2, 2] **5**
9 **5** [4, 2, 2, 2, 3] **2** [2, 6, 2, 2] **5**
10 **5** [3, 10] **2** [2, 2, 2, 2, 2, 2, 2, 2, 3, 2] **5**
11 **5** [3, 3, 4] **2** [2, 2, 3, 3, 2] **5**
12 **5** [3, 2, 4, 2] **2** [3, 2, 4, 2] **5**
13 **5** [3, 2, 2, 2, 4] **2** [2, 2, 6, 2] **5**
14 **5** [3, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2] **2** [15, 2] **5**
15 **5** [2, 15] 2 [2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 3] **5**
16 **5** [2, 6, 2, 2] **2** [4, 2, 2, 2, 3] **5**
17 **5** [2, 4, 2, 3] **2** [2, 4, 2, 3] **5**
18 **5** [2, 3, 3, 2, 2] **2** [4, 3, 3] **5**
19 **5** [2, 3, 2, 2, 2, 2, 2, 2, 2, 2] **2** [10, 3] **5**
20 **5** [2, 2, 6, 2] **2** [3, 2, 2, 2, 4] **5**
21 **5** [2, 2, 3, 3, 2] **2** [3, 3, 4] **5**
22 5 [2, 2, 2, 8] 2 [2, 2, 2, 2, 2, 2, 5] 5
23 **5** [2, 2, 2, 3, 2, 2, 2, 2] **2** [6, 5] **5**
24 **5** [2, 2, 2, 2, 3, 2, 2, 2] **2** [5, 6] **5**
25 **5** [2, 2, 2, 2, 2, 2, 5] **2** [2, 2, 2, 8] **5**
26 **5** [2, 2, 2, 2, 2, 2, 2, 2, 3, 2] **2** [3, 10] **5**
27 **5** [2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 3] **2** [2, 15] **5**
28 **5** [2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2] **2** [29] **5**

2. SINGULAR AND NONSINGULAR ALGEBRAIC SURFACES

In this section, we collect all the definitions and basic properties for working with singular and nonsingular algebraic surfaces. In particular, we introduce the main singularities for this text. A standard book for learning the basics of nonsingular complex surfaces is [B1].

2.1. **Generalities on surfaces and singularities.**

Our ground field is $\mathbb{C}$. A *surface* is a normal irreducible variety of dimension 2. Sometimes projective, sometimes quasi-projective, sometimes an analytic germ using the induced complex topology.

If the surface X is locally defined in $\mathbb{C}^n$ by its ideal $(f_1, \ldots, f_r)$, then $x \in X$ is *nonsingular* if and only if the rank of the matrix $(\partial f_i/\partial x_j)_{i,j}$ at x is equal to $n-2$. As we require that X is normal, we have that singular points are finitely many.

Let $\mathcal{O}_X$ be the structure sheaf of X (or sheaf of regular functions on X). Then we have local rings $\mathcal{O}_{X,x}$ for each point $x \in X$, and let m_x be their maximal ideals. The *Zariski cotangent space* is m_x/m_x^2 (over $\mathbb{C}$) (the tangent space is its dual). This is a $\mathbb{C}$-vector space of dimension greater than or equal to 2, and we have equality if and only if x is a nonsingular point, which is equivalent to $\mathcal{O}_{X,x}$ being a regular local ring. The *multiplicity* of $x \in X$ is the multiplicity of m_x in $\mathcal{O}_{X,x}$.

Example 2.1. Let $n > 1$ be an integer. The affine surface $X = \{z^n = xy\} \subset \mathbb{C}^3$ is singular only at $x = (0,0,0)$. It has multiplicity 2. It is called A_{n-1}, and is part of the Du Val singularities. In the following, we elaborate more on these singularities.

By definition, two singularities are isomorphic if the completion of their local rings are. By [A3, Corollary 1.6], we know that this is equivalent to the existence of analytically isomorphic neighborhoods (as stated in Artin's paper [A3], for our case this was known by Hironaka and Rossi 1964).

Remark 2.2. Let $(x \in X)$ be a singularity. The *embedding dimension* of $(x \in X)$ is the smallest dimension of any higher dimensional smooth germ $(z \in Z)$ such that we have embedding of germs $(x \in X) \subset (z \in Z)$. We have that the dimension of $(z \in Z)$ is greater than or equal to 3. The embedded dimension is the dimension of the tangent space of $(x \in X)$.

A germ of a surface singularity will be denoted by $(x \in \bar{X})$. Let

$$\pi \colon X \to \bar{X}$$

be a proper birational morphism which is a *resolution* of $(x \in \bar{X})$, and so X is nonsingular and outside of $\mathrm{Exc}(\pi) := \pi^{-1}(x)$ the morphism is an isomorphism. It is a minimal resolution if there are no (-1)-curves in $\mathrm{Exc}(\pi)$ (they always exist and are unique).

The singularity $(x \in \bar{X})$ is said to be *rational* if $R^1\pi_*\mathcal{O}_X = 0$ (One can prove that it does not depend on the chosen resolution). Hence X and $\bar{X}$ share the same irregularity and geometric genus.

Remark 2.3. Let $\bar{X}$ be projective. In general, the *geometric genus of the singularity* is defined as

$$p_g(x \in \bar{X}) := \dim_{\mathbb{C}} H^0(\bar{X}, (R^1\pi_*\mathcal{O}_X)_x).$$

We have $\pi_*\mathcal{O}_X = \mathcal{O}_{\bar{X}}$ since $\bar{X}$ is normal, and so we have from the five term Leray spectral sequence

$$0 \to H^1(\bar{X}, \mathcal{O}_{\bar{X}}) \to H^1(X, \mathcal{O}_X) \to H^0(\bar{X}, R^1\pi_*\mathcal{O}_X) \to H^2(\bar{X}, \mathcal{O}_{\bar{X}}) \to H^2(X, \mathcal{O}_X) \to 0.$$

In this way, we have

$$\chi(\mathcal{O}_{\bar{X}}) = \chi(\mathcal{O}_X) + \sum_{x \in X} p_g(x \in X).$$

For the proof, it is used the seven-term Leray spectral sequence and that

$$H^0(R^2\pi_*\mathcal{O}_X) = H^1(R^1\pi_*\mathcal{O}_X) = 0,$$

as $R^1\pi_*\mathcal{O}_X$ is supported only at points (the singular points). In [A4, Lemma 2.2], it is shown that for any positive cycle $Z = \sum_{x\in\bar{X}} Z_x$ supported on the exceptional divisor we have

$$\chi(\mathcal{O}_{\bar{X}}) - \chi(\mathcal{O}_X) = \sum_{x \in X} p_g(x \in X) \geq \sum_{x \in X} p_a(Z_x),$$

where $p_a(Z_x) = 1 - \chi(Z_x)$ is the arithmetic genus of Z_x.

Remark 2.4. (Intersection theory on surfaces) For a smooth projective surface X, we have an intersection theory; cf. [B1]. Mumford [M5, p.6] showed that the intersection matrix of $\mathrm{Exc}(\pi)$ is negative definite over any singularity. Mumford also worked out in [M5] an intersection theory for Weil divisors on a normal projective surface. All of this will be used throughout this text. A curve C on a nonsingular surface such that $C^2 = -m$ and $C \simeq \mathbb{P}^1$ is called a $(-m)$-*curve*.

Artin [A5] studies equivalent conditions to be a rational singularity in terms of $\mathrm{Exc}(\pi) = \sum_i E_i$. He studies the cohomology of the schemes $Z := \sum_i r_i E_i$ (where $r_i \geq 0$), showing that rationality holds if and only if $p_a(Z) \leq 0$ for all such Z. (This uses $\varprojlim_{(r)\to\infty} H^1(\mathrm{Exc}(\pi), \mathcal{O}_{Z_{(r)}}) = R^1\pi_*\mathcal{O}_X$.) He goes further to define the fundamental cycle.

Definition 2.5. The *fundamental cycle* $Z = \sum_i r_i E_i$ is the unique smallest effective Z such that $Z \cdot E_i \leq 0$ for all i.

In the case of a singularity and to make it unique with respect to the singularity, we choose the fundamental cycle of the unique *minimal resolution*, i.e. the resolution where no E_i is a (-1)-curve.

Proposition 2.6. *Let Z be the fundamental cycle of $(x \in \bar{X})$.*

- *We have $p_a(Z) \geq 0$, and $p_a(Z) = 0$ if and only if $(x \in \bar{X})$ is rational.*
- *If $(x \in \bar{X})$ is rational, then $\mathrm{Exc}(\pi)$ is a tree of smooth rational curves, $-Z^2$ is the multiplicity, and $-Z^2 + 1$ is the dimension of the tangent space at x.*
- *The determinant of $(E_i \cdot E_j)$ is the torsion group of $H_1(L, \mathbb{Z})$, where L is the link of the singularity $(x \in \bar{X})$ (see [M5]).*

Proposition 2.7 (Artin contractibily theorem)**.** [A4, Theorem 2.3] *Let $\mathrm{Exc} := \sum_i E_i$ be a connected collection of curves in a normal surface X. The following are equivalent:*

(a) Exc *is contractible and if $\pi\colon X \to \bar{X}$ is the contraction then $\chi(\mathcal{O}_{\bar{X}}) = \chi(\mathcal{O}_X)$.*
(b) – *The intersection matrix of* Exc *is negative definite.*
  – *For every cycle $Z > 0$ with support in* Exc *we have $p_a(Z) \leq 0$.*
(c) – *The intersection matrix of* Exc *is negative definite.*
  – *The fundamental cycle Z satisfies $p_a(Z) = 0$ (see* [A5, Theorem 3]*).*

Moreover, if X is projective and (a) holds, then $\bar{X}$ is also projective.

Remark 2.8. In [H6, Example 5.7.3] an example is shown (due to Hironaka) starting with a smooth plane cubic and 10 linearly independent points in its group law. Blow them up to obtain an uncontractible smooth curve of self-intersection -1. It can not be contracted to an algebraic surface, but it can be contracted to an analytic singular surface.

Example 2.9. Let $\{E_1, \ldots, E_r\}$ be a negative definite chain of nonsingular rational curves in X. Let us assume that it is minimal, that is, $E_i = -e_i \leq -2$. Then, the associate fundamental cycle is $Z = \sum_i E_i$. One can check that $h^1(\mathcal{O}_Z) = 0$, and so $p_a(Z) = 0$. The intersection matrix

$$\begin{bmatrix} -e_1 & 1 & & & \\ 1 & -e_2 & 1 & & \\ & 1 & -e_3 & & \\ & & & \ddots & 1 \\ & & & 1 & -e_r \end{bmatrix} \tag{2.1}$$

which is indeed negative definite (induction on principal minors). Hence we can contract this chain into a rational singularity. We have that its multiplicity is $-Z^2 = \sum_{i=1}^r (e_i - 2) + 2$, and its embedded dimension is $-Z^2 + 1 = \sum_{i=1}^r (e_i - 2) + 3$. We will see later that this singularity is isomorphic to a cyclic quotient singularity (which is the same as toric singularity in dimension 2), and the resolution can be constructed using toric methods. The order of the quotient is the absolute value of the determinant of the intersection matrix.

A very relevant equation is the relation of the canonical classes in the minimal resolution $\pi\colon (\sum_i E_i \subset X) \to (x \in \bar{X})$, namely

$$K_X \equiv \pi^*(K_{\bar{X}}) + \sum_i d_i E_i,$$

where the $d_i = d_i(E_i)$ are the *discrepancies* of $\mathrm{Exc}(\pi)$. They are uniquely determined. As it is minimal, we have $d_i \leq 0$ for all i (see [KM1, Corollary 4.2]).

- If $d_i = 0$ for all i, then $x \in \bar{X}$ is called *canonical*. Important in canonical models of surfaces of general type.
- If $-1 < d_i \leq 0$, then $x \in \bar{X}$ is called *log terminal*. They are all *quotient singularities* by Kawamata, that is, there is a finite group G acting on $\mathbb{C}^2$ such that $(x \in \bar{X})$ is isomorphic to the quotient $\mathbb{C}^2/G$ at $(0,0)$.
- If $-1 < d_i \leq 0$, then $x \in \bar{X}$ is called *log canonical*. Important for singularities in Kollár–Shepherd-Barron–Alexeev surfaces which compactify the moduli space of surfaces of general type [KSB].

Example 2.10. A *Du Val singularity* (also known as rational double points or ADE singularities) is a rational singularity $(x \in \bar{X})$ which has multiplicity 2. Thus we have $-Z^2 = 2$ and $p_a(Z) = 0$, where Z is the fundamental cycle. Thus $K_X \cdot Z = -Z^2 - 2 = 0$ and so $0 = (\sum_i d_i E_i) \cdot Z$. On the other hand, since the E_i are smooth rational curves, we have $K_X \cdot E_i = e_i - 2 \geq 0$. Therefore, $d_i = 0$ for all i, and the E_i are (-2)-curves. One can classify the minimal resolution diagrams as the Dynkin diagrams A_n (this is a chain as above), D_n, E_6, E_7, and E_8. This and rational triple points are described in [A5], see Figure 7. Du Val singularities are also called Kleinian singularities (from quotients of the 2-sphere S^2 by finite groups (equivalently finite groups of $\mathrm{Aut}(\mathbb{P}^1_{\mathbb{C}})$)), and also canonical surface singularities (just get the above by assuming $d_i = 0$ for all i). Local models in $(0 \in \mathbb{C}^3)$ for ADE singularities are:

$$A_n\ (n \geq 1) : z^2 + x^2 + y^{n+1} = 0 \qquad D_n\ (n \geq 4) : z^2 + y(x^2 + y^{n-2}) = 0$$
$$E_6 : z^2 + x^3 + y^4 = 0 \qquad E_7 : z^2 + x(x^2 + y^3) = 0 \qquad E_8 : z^2 + x^3 + y^5 = 0.$$

In particular, any Du Val singularity can be seen as a branched double cover and as a quotient singularity.

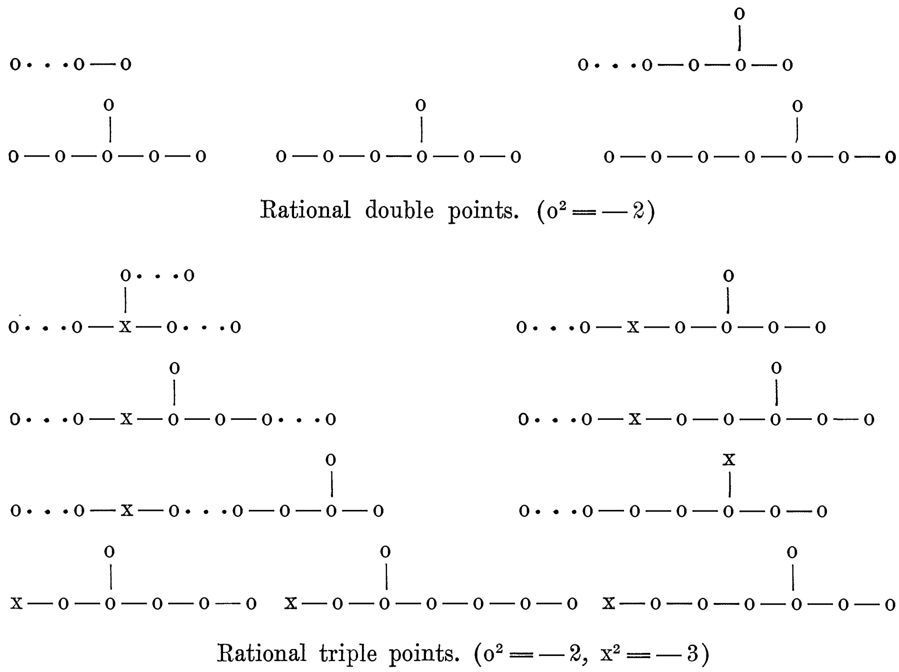


FIGURE 7. Classification of double and triple rational points via their resolution graphs (from Artin's article [A5]). See Remark 2.13.

Exercises.

(1) You can resolve each Du Val singularity by blowing up over the singularity of the branched plane curve and then taking the double cover. For example, for A_n we blow-up over the singularity of $\{x^2 + y^{n+1} = 0\}$ and their pull-backs, until its pull-back has only disjoint curves with odd multiplicities. Then take the double cover branched along these odd curves, and obtain a resolution for A_n. Do this for as many Du Val singularities as you can.

(2) Consider the surface of cuboids, related to the famous perfect cuboid problem, defined in $\mathbb{P}^6_{\mathbb{C}}$ by the equations $x_3^2 = x_0^2 + x_1^2 + x_2^2$, $x_4^2 = x_0^2 + x_1^2$, $x_5^2 = x_0^2 + x_2^2$, and $x_6^2 = x_1^2 + x_2^2$. Show that it has 48 singularities, and each of them is of type A_1.

(3) Verify that the fundamental cycle Z in Example 2.9 is indeed $\sum_{i=1}^r E_i$. Then use Proposition 2.7 to show that it is contractible to a singularity. Then use Proposition 2.6 in the formulas for multiplicity and embedded dimension.

(4) As you know, a finitely generated $\mathbb{C}$-algebra A corresponds to an affine variety. Indeed, it is defined by the zero set of the corresponding kernel of the surjective morphism $\mathbb{C}[x_1, \ldots, x_m] \to A$. Given a finite group G acting on $\mathbb{C}^m$ by isomorphisms

fixing $(0,0)$, you can consider the $\mathbb{C}$-algebra $\mathbb{C}[x_1,\ldots,x_m]^G$ of polynomials invariants by G. Show that it is finitely generated and normal. The corresponding variety is, by definition, $\mathbb{C}^m/G$. (It could be nonsingular too.) For example, show that $(x,y)\mapsto(-x-y)$ is a $\mathbb{Z}/2$ action on $\mathbb{C}^2$ such that $\mathbb{C}^2/(\mathbb{Z}/2)=\{x^2+y^2+z^2=0\}\subset\mathbb{C}^3$.

(5) Any Du Val singularity is a quotient singularity. Find all finite actions on $\mathbb{P}^1$ (two times two invertible matrices), and quotient $\mathbb{C}^2$ by them. Those are the Du Val singularities. Compute some.

(6) Find an action on $\mathbb{C}^2$ by the dihedral group such that the quotient is again $\mathbb{C}^2$.

2.2. **Cyclic quotient singularities.**

Definition 2.11. Let $m>1$ be an integer and let μ be an mth primitive unit of 1. Let $a_i>0$ be integers coprime to m. Consider the action $\mathbb{C}^d\to\mathbb{C}^d$ of $\mathbb{Z}/m$ given by

$$T(x_1,\ldots,x_d)=(\mu^{a_1}x_1,\ldots,\mu^{a_d}x_d).$$

A *cyclic quotient singularity* (c.q.s.) is a singularity isomorphic to the germ at zero of $\mathbb{C}^d/\langle T\rangle$ for some m and a_is. See Exercise §2.14. The notation will be $\frac{1}{m}(a_1,\ldots,a_d)$.

If $d=1$, then quotients are nonsingular. If $d=2$, then we can restrict ourselves to $1/m(1,q)$ where $0<q<m$ is coprime to m. These are called *Hirzebruch-Jung singularities* [BHPVdV, III.5]. For $d\geq 3$ these singularities are rigid, which is a result of Schlesinger [S3] [5]. Therefore, only in dimension two we may have nonrigid quotient singularities. In fact they are non rigid and their nontrivial deformation theory is key for what follows.

Let $0<q<m$ be coprime integers. The action $(x,y)\mapsto(\mu x,\mu^q y)$ on $\mathbb{C}^2$ induces an action on $\mathbb{C}[x,y]$, and by definition, the c.q.s. is defined as the germ corresponding to the variety associated with $\mathbb{C}[x,y]^{\mathbb{Z}/m}$. We have the inclusions of finitely generated $\mathbb{C}$-algebras

$$\mathbb{C}[x^m,y^m]\subset\mathbb{C}[x^m,y^m,x^{m-q}y]\subset\mathbb{C}[x,y]^{\mathbb{Z}/m}\subset\mathbb{C}[x,y],$$

which translates into the diagram in Figure 8, where $u=x^m, v=y^m, w=x^{m-q}y$.

Figure 8 also includes the minimal resolution $\phi\colon X\to\bar{X}:=\frac{1}{m}(1,q)$, which can be done through standard toric methods (subdivision of the cone corresponding to $\mathbb{C}[x,y]^{\mathbb{Z}/m}$ as in Figure 8). It turns out that the exceptional divisors E_i are $(-e_i)$-curves forming a chain, where the e_i are computed through the Hirzebruch-Jung continued fraction $\frac{m}{q}=[e_1,\ldots,e_r]$. This is the connection with the previous section!

Definition 2.12. The continued fraction $\frac{m}{q}=[e_1,\ldots,e_r]$ defines the sequence of integers

$$0=\beta_{r+1}<1=\beta_r<\ldots<q=\beta_1<m=\beta_0$$

where $\beta_{i+1}=e_i\beta_i-\beta_{i-1}$. In this way, $\frac{\beta_{i-1}}{\beta_i}=[e_i,\ldots,e_r]$. Partial fractions $\frac{\alpha_i}{\gamma_i}=[e_1,\ldots,e_{i-1}]$ are computed through the sequences

$$0=\alpha_0<1=\alpha_1<\ldots<q^{-1}=\alpha_r<m=\alpha_{r+1},$$

[5] More precisely: If a finite group G acts on an affine scheme Y, smooth over a field k, with a single fixed point $y\in Y$, then the quotient scheme $X=Y/G$ has an isolated singularity at the point $x\in X$ under y, and this singularity is rigid, provided $\dim(Y)\geq 3$ and the order of G is not divisible by the characteristic of k.

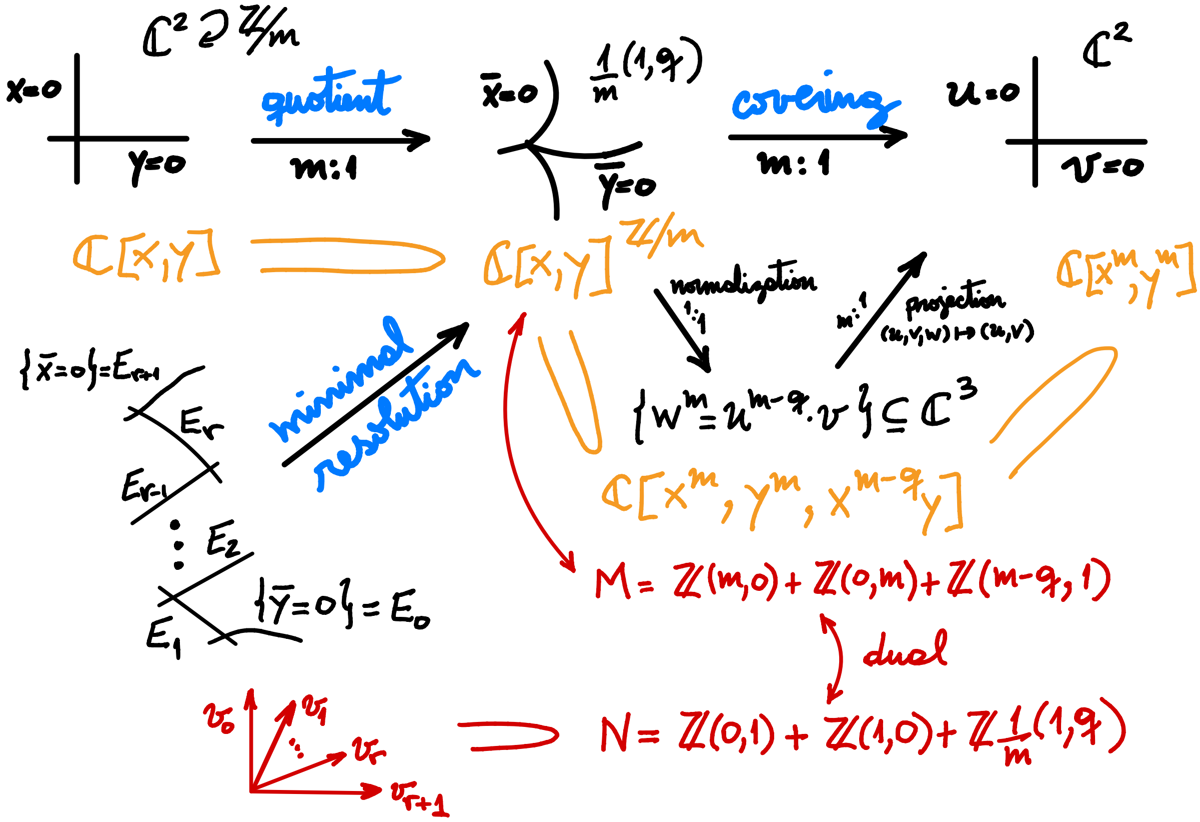


FIGURE 8. The trilogy: quotient, covering, minimal resolution.

where $\alpha_{i+1} = e_i\alpha_i - \alpha_{i-1}$, and $\gamma_0 = -1$, $\gamma_1 = 0$, $\gamma_{i+1} = e_i\gamma_i - \gamma_{i-1}$. We have $\alpha_{i+1}\gamma_i - \alpha_i\gamma_{i+1} = -1$, and $\beta_i = q\alpha_i - m\gamma_i$.

These numbers appear in the pull-back formulas

$$\phi^*\big((\bar{y}=0)\big) \equiv \sum_{i=0}^{r+1} \frac{\beta_i}{m} E_i, \qquad \phi^*\big((\bar{x}=0)\big) \equiv \sum_{i=0}^{r+1} \frac{\alpha_i}{m} E_i,$$

and

$$K_X \equiv \phi^*(K_{\bar{X}}) + \sum_{i=1}^{r} \big(-1 + \frac{\beta_i+\alpha_i}{m}\big) E_i.$$

Here E_0 and E_{r+1} are the proper transforms of $(\bar{y}=0)$ and $(\bar{x}=0)$ respectively.

Remark 2.13. It turns out that a singularity whose resolution is a chain of $\mathbb{P}^1$ can be characterized by the information of its dual graph. In general, given a minimal resolution with simple normal crossings, the *weighted dual graph* of $\mathrm{Exc}(\pi)$ is the graph whose vertices represent the curves E_i, the edges represent the intersections between curves, and for each vertex we associate the genus and the self-intersection of the corresponding E_i. If the genus is 0, then we do not write it. A singularity $(x \in \bar{X})$ is said to be *taut* if given a singularity $(x' \in \bar{X}')$ with the same weighted dual graph we have $(x \in \bar{X}) \simeq (x' \in \bar{X}')$ (see [L1], [L2]). Particularly see [L1, Theorem 6.20]. Laufer [L2] classifies taut singularities, in particular c.q.s. are taut and so they depend only on their HJ continued fraction.

The equations for $\bar{X}$ can be described by means of the extended Riemenschneider 2×2 minors. For that, we need to use the *dual* H-J continued fraction

$$\frac{m}{m-q} = [b_1, \ldots, b_s].$$

One can give the following presentation to the $\mathbb{C}$-algebra of $\mathbb{C}^2/(\mathbb{Z}/m)$:

$$\mathbb{C}[z_0, z_1, \ldots, z_s, z_{s+1}]/(z_i z_j - z_{j-1} z_{j-1}^{b_{j-1}-2} \cdots z_{i+1}^{b_{i+1}-2} z_{i+1} : \text{ for } 0 \leq i < j-1 \leq s).$$

Therefore the embedded dimension of $1/m(1,q)$ is $\leq s+2$, but one can easily show that it is $s+2$ because by the Riemenschneider's dot diagram we have

$$\sum_{i=1}^{r} (e_i - 2) + 3 = s + 2.$$

Note that the multiplicity of $\mathbb{C}^2/(\mathbb{Z}/m)$ is then $s+1$.

Example 2.14. Say $q = 1$, and so $\frac{m}{1} = [m]$ and $\frac{m}{m-1} = [\underbrace{2, \ldots, 2}_{m-1}]$. Then the $\mathbb{C}$-algebra is

$$\mathbb{C}[z_0, \ldots, z_m]/(z_i z_j - z_{j-1} z_{i+1})$$

for $0 \leq i < j-1 \leq m-1$. Thus, this is the cone over the rational normal curve of degree m (Veronese embeddings of $\mathbb{P}^1$!). For example, for $m = 1$ this is $\{z_0 z_2 = z_1^2\} \subset \mathbb{C}^3$. On the other extreme, if $q = m-1$, then the dual fraction is $\frac{m}{1} = [m]$, and so we obtain as $\mathbb{C}$-algebra

$$\mathbb{C}[z_0, z_1, z_2]/(z_0 z_2 - z_1^m),$$

and these are the A_{m-1} Du Val singularities.

Definition 2.15. Let $(x \in \bar{X})$ be a normal surface singularity. Assume that $K_{\bar{X}}$ is $\mathbb{Q}$-Cartier. The *index* of $(x \in \bar{X})$ is the smallest integer n such that $nK_{\bar{X}}$ is a line bundle.

Lemma 2.16. *The index of* $(x \in \bar{X}) = \frac{1}{m}(1,q)$ *is* $\frac{m}{gcd(m,q+1)}$.

Proof. Consider the minimal resolution $\pi \colon X \to \bar{X}$. We have

$$K_X \equiv \pi^*(K_{\bar{X}}) + \sum_{i=1}^{r} \big(-1 + \frac{\beta_i + \alpha_i}{m} \big) E_i$$

with the notation above. For a fixed i, one sees (from the recursion formulas for β_i and α_i) that gcd$(m, \alpha_i + \beta_i)$ divides $\alpha_j + \beta_j$ for all j. Set $i = 1$. (Precisely when $q = m-1$ the discrepancies are zero, so the index of $\bar{X}$ is one.) □

Exercises.

(1) Let μ be an mth primitive root of 1, and let a, b be integers coprime to m. We have the action $T(x,y) = (\mu^a x, \mu^b y)$ on $\mathbb{C}^2$. Show that there is $0 < q < m$ such that $\frac{1}{m}(a,b) := \mathbb{C}^2/\langle T \rangle$ is isomorphic to $\frac{1}{m}(1,q)$.
(2) Let μ be an mth primitive root of 1, and consider the action $T(x,y) = (\mu x, \mu^q y)$ on $\mathbb{C}^2$ where gcd$(q,m) = d$. Find $0 < q' < m'$ such that $\frac{1}{m'}(1,q') = \mathbb{C}^2/\langle T \rangle$.
(3) Let a, b, c be positive integers and pairwise coprime. Show that the weighted projective plane $\mathbb{P}(a,b,c)$ has precisely 3 singularities: $\frac{1}{c}(a,b)$, $\frac{1}{a}(b,c)$ and $\frac{1}{b}(c,a)$.

(4) Evaluate Figure 8 for $m = 19$ and $q = 7$, finding in particular the subdivision that resolves the singularity, and the α_i, β_i and discrepancies.
(5) Show that $\frac{1}{19}(1,7)$ is isomorphic to $\frac{1}{19}(1,11)$, but is not isomorphic to $\frac{1}{19}(1,12)$.
(6) Show that indeed $\frac{1}{m}(1,q)$ is the normalization of $\{w^m = u^{m-q}v\} \subset \mathbb{C}^3$ (Figure 8).
(7) Consider the surfaces

$$S_1 = \{(x,y,z) \in \mathbb{C}^3 : z^m = x^a y^b\} \text{ and } S_2 = \{(x,y,z) \in \mathbb{C}^3 : z^m = x^{m-q}y\}$$

with the condition $a + bq \equiv 0 (\mathrm{mod}\, m)$ and a, b coprime to m. Show that the normalizations of S_1 and S_2 are isomorphic.
(8) Let $0 < q < m$ be coprime integers. Consider the minimal resolution X of the normalization $\bar{X}$ of the mth cyclic cover of $\mathbb{P}^2$ branched along $\{xy^q z^{m-q-1} = 0\}$.
    (a) Show that X is the Hirzebruch surface $\mathbb{F}_m$ when $q = m - 1$.
    (b) Find the singularities of $\bar{X}$ for any fixed pair q, m.
    (c) Show that X is a rational surface.
    (d) Thus, we have $\chi(\mathcal{O}_X) = 1$. Compute in another way $\chi(\mathcal{O}_X)$ and use that to find some equation which involves the Dedekind sum $s(q,m)$ (see Remark 1.9).

2.3. **T-singularities.**

Something bizarre as a presentation (see Exercise §1.1(6)).

Lemma 2.17. *Let $0 < a < n$ be integers with $\gcd(a,n) = 1$. Let*

$$n/(n-a) = [x_1, \ldots, x_p] \quad \text{and} \quad n/a = [y_1, \ldots, y_q].$$

Then for any $d \geq 1$, we have $\frac{dn^2}{dn^2-(dna-1)} = [x_1, \ldots, x_p, 1+d, y_q, \ldots, y_1]$. Moreover its dual is

$$\frac{dn^2}{dna-1} = [y_1, \ldots, y_{q-1}, y_q + 1, 2, \ldots, 2, x_p + 1, x_{p-1}, \ldots, x_1],$$

for $d > 1$, where we have $d-2$ 2s in the middle, or $\frac{n^2}{na-1} = [y_1, \ldots, y_{q-1}, y_q + x_p, x_{p-1}, \ldots, x_1]$.

Proof. See [HP, Lemma 8.5] or [PSU2, Corollary 2.1 and 2.2] or prove it yourself using Section 1. □

The c.q.s. $\frac{1}{dn^2}(1, dna-1)$ with $0 < a < n$ and $\gcd(a,n) = 1$ together with Du Val singularities form the important class of *T-singularities*.

They were first introduced by Kollár–Shepherd-Barron as the quotient singularities that admit a $\mathbb{Q}$-Gorenstein one-parameter smoothing [KSB, Definition 3.7]. These smoothings will be discussed later, but this roughly means the existence of a 3-fold germ over a disk, so that one fiber is the T-singularity, all other fibers are smooth, and the canonical class of the 3-fold is $\mathbb{Q}$-Cartier. An example is the node $A_1 = 1/2(1,1)$ taking the 3-fold $\{x^2 + y^2 + z^2 = t\} \subset \mathbb{C}^3 \times \mathbb{D}$ with the projection into the disk $\mathbb{D} := \{t \in \mathbb{C} \colon |t| < 1\}$. (We do not care about the radius here, our disks will always be just analytic germs of nonsingular curves.)

Among T-singularities, we have the key *Wahl singularities*, which are the $1/n^2(1, na-1)$ with $0 < a < n$ and $\gcd(a,n) = 1$.

Lemma 2.18. *Let $(x \in \bar{X}) = \frac{1}{m}(1,q)$ be a cyclic quotient singularity. The following are equivalent*

(1) $m = dn^2$ *and* $q = dna - 1$ *with gcd*$(n, a) = 1$.
(2) $K_{\bar{X}}^2$ *is an integer.*
(3) $\frac{m}{gcd(m,q+1)}$ *divides gcd*$(m, 1+q)$.

Proof. If $\pi\colon X \to \bar{X}$ is the minimal resolution, then we have

$$(K_X - \pi^* K_{\bar{X}})^2 = \sum_{i=1}^{r}(2 - e_i) - \frac{q + q' + 2}{m} + 2$$

so $q + q' + 2 \equiv 0(\text{mod } m)$, where $0 < q' < m$ and $qq' \equiv 1(\text{mod } m)$. This gives (1) if and only if (2). (1) if and only if (3) is trivial. □

The c.q.s. that are T-singularities can also be characterized by their ADE graphs or the T-chains, which were studied in Section 1.2 via the Wahl algorithm ([KSB, Proposition 3.11]).

Corollary 2.19. *If* $\frac{dn^2}{dna-1} = [e_1, \ldots, e_r]$, *then*

$$r - d + 2 = \sum_{i=1}^{r}(e_i - 2).$$

If $\pi\colon X \to \bar{X}$ *is the minimal resolution, then* $K_X^2 + r - d + 1 = K_{\bar{X}}^2$.

As we saw before, Wahl singularities are key for working with T-singularities. If $\frac{n^2}{na-1} = [e_1, \ldots, e_r]$, then

$$\frac{dn^2}{dna - 1} = [e_1, \ldots, e_r, 1, e_1, \ldots, e_r, 1, \ldots, 1, e_1, \ldots, e_r]$$

where the numbers of 1s is $d - 1$. Hence any non-Du Val T-singularity has a partial resolution consisting of $d-1$ Wahl singularities of the same type joined by $d-1$ rational curves whose intersection with canonical class is trivial. This is called *M-resolution* [BC2]. We will elaborate more on them later.

Let $[e_1, \ldots, e_r]$ be a Wahl chain. We define integers $\delta_1, \ldots, \delta_r$ in the following inductive way. If $r = 1$ then $\delta_1 := 1$. If we already defined $\delta_1, \ldots, \delta_r$ for $[a_1, \ldots, a_r]$, then we assign

$$\delta_1, \ldots, \delta_r, \delta_1 + \delta_r \text{ to } [e_1 + 1, \ldots, e_r, 2]$$
$$\delta_1 + \delta_r, \delta_1, \ldots, \delta_r \text{ to } [2, e_1, \ldots, e_r + 1].$$

These numbers compute the discrepancies. If $\frac{n^2}{na-1} = [e_1, \ldots, e_r]$ has numbers $\delta_1, \ldots, \delta_r$, then

$$K_X \equiv \pi^* K_{\bar{X}} + \sum_{i=1}^{r}\left(-1 + \frac{\delta_i}{\delta_1 + \delta_r}\right) E_i. \tag{2.2}$$

We note that $\delta_1 = a$ and $\delta_1 + \delta_r = n$. We have a similar description for any T-singularity.

The index of the T-singularity $\frac{1}{dn^2}(1, dna - 1)$ is n, and it satisfies (see Exercise §1.2 5)

$$n \leq F_{r-d}$$

where F_i is the ith Fibonacci number defined by the recursion $F_{-2} = 1$, $F_{-1} = 1$, and

$$F_i = F_{i-1} + F_{i-2}$$

for $i \geq 0$. We have equality (Fibonacci T-singularity) if and only if the Hirzebruch-Jung continued fraction has the form

$$[3, \ldots, 3, 2, \ldots, 2, 4, 3, \ldots, 3, 2] \text{ or } [3, \ldots, 3, 5, 3, \ldots, 3, 2].$$

A projective surface $\bar{X}$ with only T-singularities satisfies the Noether's formula

$$K_{\bar{X}}^2 + \chi_{\text{top}}(\bar{X}) + \sum_{x \in \text{Sing}(\bar{X})} \mu_x = 12\chi(\mathcal{O}_{\bar{X}}),$$

where μ_p is the Milnor number of (a $\mathbb{Q}$-Gorenstein smoothing of) x (see [HP, p. 172]). We have the Milnor numbers:

- If x is rational double point of type A_d, D_d, or E_d, then $\mu_x = d$.
- If x is of type $\frac{1}{dn^2}(1, dna-1)$, then $\mu_x = d-1$.

One can find a better bound for the index n via some bound for $r-d$. For example, if $\bar{X}$ is a nonrational projective surface with one T-singularity and $K_{\bar{X}}$ is ample, then we have the following theorem [RU1] (see also [ES]).

Theorem 2.20. *Let S be the minimal model of X (resolution of $\bar{X}$), and let $\kappa(S)$ be the Kodaira dimension of S.*

1. *If $\kappa(S) = 0$, then $r - d \leq 4K_{\bar{X}}^2$.*
2. *If $\kappa(S) = 1$, then $r - d \leq 4K_{\bar{X}}^2 - 2$.*
3. *If $\kappa(S) = 2$, then*

$$r - d \leq 4(K_{\bar{X}}^2 - K_S^2) - 4$$

when $K_{\bar{X}}^2 - K_S^2 > 1$, $r - d \leq 1$ otherwise.

Those inequalities are optimal, and equality can be classified. The integer d can be bounded using the log BMY inequality, and so we are really bounding r. See details in [RU1]. For two or more singularities see the recent pre-print [FRU]. In that work, we have that $\bar{X}$ has l singularities $\frac{1}{d_j n_j^2}(1, d_j n_j a_j - 1)$ with ample canonical class, $X \to \bar{X}$ is the minimal resolution with C equals the exceptional divisor, and $\pi \colon X \to S$ is the composition of blow-downs into a minimal model S. Then either we have

$$\sum_{j=1}^{l} (r_j - d_j) \leq 2(K_{\bar{X}}^2 - K_S^2) - K_S \cdot \pi(C),$$

or there are particular configurations E_i that are exceptional for π such that $E_i \cdot C = 1$. We classify all of these special E_i. It is given a bound when $\bar{X}$ is not rational. The paper of Rana-Urzúa [RU1] bounds the case of one T-singularity in a rational surface, but it involves a degree that could be artificially arbitrarily large. We know by Alexeev's boundedness that the rational surface's case is bounded after we fix K^2, but it **remains open to find an optimal bound**. Based only on examples, an optimal bound could be close to $4K^2 + 6$.

There are many examples of $\bar{X}$ with only T-singularities. Just via blow-ups on appropriate configurations of rational curves. In general, they will have $K_{\bar{X}}$ not even big. To have a big and nef canonical class is more subtle, but again there are plenty of examples. This was started with the rational blowdown technique [FS]. We compute one in the next example.

Example 2.21. On the left of Figure 9 we have a K3 surface S with a configuration of six (-2)-curves that intersect transversally as shown. On the right, we have a particular composition of six blow-ups X. We find in this way two Wahl chains $[2,2,4,5,3,2,4]$ and $[4]$. We contract them $\phi\colon X \to \bar{X}$ by Artin's theorem. One can show that $K_{\bar{X}}$ is ample for a general choice of such S. We compute $K_{\bar{X}}^2 = 2$.

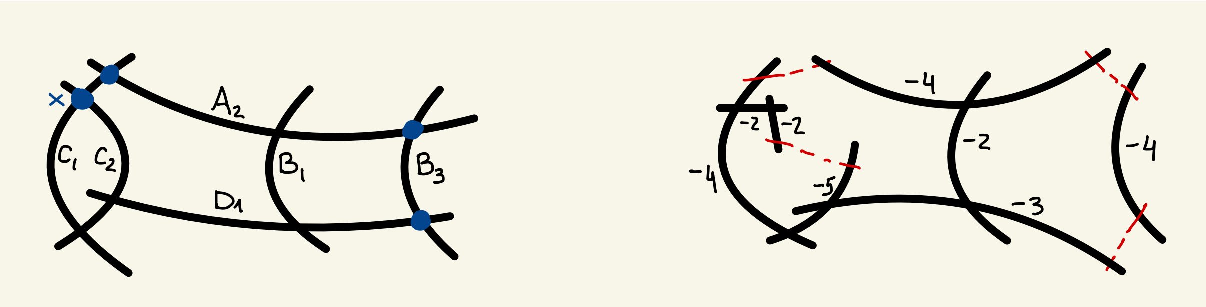


FIGURE 9. An example of a surface $\bar{X}$ with $K_{\bar{X}}$ is ample [RU2].

Exercises.

(1) Let (a,b,c) be a Markov triple. Show that $\mathbb{P}(a^2,b^2,c^2)$ has three Wahl singularities.

(2) Show that a projective surface $\bar{X}$ with only T-singularities satisfies the Noether's formula

$$K_{\bar{X}}^2 + \chi_{\text{top}}(\bar{X}) + \sum_{x\in \text{Sing}(\bar{X})} \mu_x = 12\chi(\mathcal{O}_{\bar{X}}),$$

where μ_p is the Milnor number of a $\mathbb{Q}$-Gorenstein smoothing of x.

(3) Show that in Example 2.21 we obtain $K_{\bar{X}}$ big and nef, just by comparing the canonical class of K_X with $K_{\bar{X}}$. Compute that $K_{\bar{X}}^2 = 2$.

(4) Show that there are infinitely many T-singulatities on surfaces $\bar{X}$ when $K_{\bar{X}}$ is only big but not nef. (This has to do with the birational geometry of the next sections.)

(5) $\star$ In [LP1] there are various examples of $\bar{X}$ with $K_{\bar{X}}$ big and nef, starting with a rational elliptic surface with sections. See also [PPS1], [PPS2], [SU1], [RU3]. Find your own example with $K_{\bar{X}}^2 = 1$.

(6) $\star\star$ It turns out that the construction of singular surfaces $\bar{X}$ with only T-singularities and $K_{\bar{X}}$ big and nef has to do directly with constructions of exotic blow-ups of $\mathbb{P}^2$ at few points (see [RU3]). We must have $0 < K_{\bar{X}}^2 < 9$, and the number of points blown-up is $9 - K_{\bar{X}}^2$. There are known examples for $K_{\bar{X}}^2 = 1,2,3,4,5$ (see previous exercise), and there are very few for $K_{\bar{X}}^2 = 5$ (the first ones are in [RU3]). Construct examples for $K_{\bar{X}}^2 \geq 6$. By other geometric means, there are exotic blow-ups of $\mathbb{P}^2$ at 3 and 2 points (very few examples! [AP1], [AP2]), but there is no example at all of an exotic blow-up of $\mathbb{P}^2$ at one point.

(7) $\star\star$ As discussed in this section, by Alexeev's boundedness there is a finite list of T-singularities for projective surface $\bar{X}$ with $K_{\bar{X}}$ big and nef and fixed $K_{\bar{X}}^2$. The results described in [RU1, FRU] give optimal ways to bound singularities when $\bar{X}$ is not rational. Find similar bounds in the case of rational surfaces.

The table shows results of computer searches by the program ComputerSearch [R1]. It is done for surfaces $\bar{X}$ with two Wahl singularities and $K^2_{\bar{X}} = 4$, starting with the rational elliptic fibration with sections that has singular fibers $I_9 + 3I_1$. Each line is one example. The (n, a) corresponds to a Wahl singularity, length is the length of the Wahl chain. GCD is $\gcd(n_1, n_2)$ for the n_i in the example.

2 chains, $K^2 = 4$									
(n,a)	Length	(n,a)	Length	GCD	Nef	$\mathbb{Q}$-ef	Obstruction 0	WH	Index
(25, 11)	7	(25, 11)	7	25	YES	YES	YES	–	1129
(31, 14)	8	(31, 13)	7	31	YES	YES	YES	–	1130
(36, 11)	8	(31, 14)	8	1	YES	YES	YES	–	1131
(39, 11)	9	(16, 3)	7	1	YES	YES	NO(2)	–	1132
(39, 11)	9	(16, 3)	7	1	YES	YES	NO(2)	NO	1133
(39, 11)	9	(27, 5)	8	3	YES	YES	YES	–	1134
(39, 11)	9	(27, 5)	8	3	YES	YES	YES	NO	1135
(41, 15)	8	(29, 13)	8	1	YES	YES	YES	–	1136
(41, 15)	8	(39, 11)	9	1	YES	YES	YES	NO	1137
(49, 22)	9	(28, 11)	8	7	YES	YES	YES	NO	1138
(61, 16)	10	(29, 11)	7	1	YES	YES	YES	NO	1139
(65, 18)	9	(17, 8)	9	1	YES	YES	YES	NO	1140
(65, 18)	9	(52, 15)	11	13	YES	YES	YES	NO	1141
(73, 21)	14	(22, 3)	9	1	YES	YES	YES	–	1142
(76, 31)	10	(11, 4)	5	1	YES	YES	YES	–	1143
(76, 29)	9	(17, 8)	9	1	YES	YES	YES	NO	1144
(79, 24)	10	(18, 7)	6	1	YES	YES	NO(2)	NO	1145
(89, 39)	11	(9, 2)	5	1	YES	YES	YES	–	1146
(89, 39)	11	(11, 2)	6	1	YES	YES	YES	NO	1147
(94, 41)	10	(37, 10)	8	1	YES	YES	YES	–	1148
(96, 17)	12	(26, 5)	9	2	YES	YES	YES	–	1149
(98, 19)	13	(19, 5)	7	1	YES	YES	YES	–	1150
(103, 27)	11	(14, 5)	6	1	YES	YES	YES	NO	1151
(107, 38)	11	(18, 7)	6	1	YES	YES	NO(2)	NO	1152
(109, 16)	13	(23, 6)	8	1	YES	YES	YES	–	1153
(113, 17)	13	(109, 16)	13	1	YES	YES	YES	NO	1154
(117, 41)	13	(13, 2)	7	13	YES	YES	YES	NO	1155
(117, 31)	11	(49, 15)	9	1	YES	YES	YES	NO	1156
(128, 37)	12	(32, 9)	8	32	YES	YES	YES	NO	1157
(128, 37)	12	(73, 21)	14	1	YES	YES	YES	NO	1158
(145, 42)	12	(23, 7)	7	1	YES	YES	NO(2)	NO	1159
(151, 45)	12	(11, 4)	5	1	YES	YES	YES	NO	1160
(153, 40)	12	(5, 1)	4	1	YES	YES	YES	–	1161
(157, 58)	11	(11, 5)	6	1	YES	YES	YES	NO	1162
(157, 28)	13	(41, 7)	11	1	YES	YES	YES	NO	1163
(163, 64)	13	(74, 29)	10	1	YES	YES	YES	NO	1164
(164, 61)	12	(4, 1)	3	4	YES	YES	NO(2)	–	1165
(169, 70)	11	(23, 7)	7	1	YES	YES	YES	–	1166
(175, 67)	11	(2, 1)	1	1	YES	YES	NO(2)	–	1167
(183, 67)	11	(7, 2)	4	1	YES	YES	NO(2)	NO	1168
(183, 38)	13	(9, 4)	5	3	YES	YES	YES	NO	1169
(187, 79)	11	(3, 1)	2	1	YES	YES	NO(2)	–	1170
(187, 79)	11	(3, 1)	2	1	YES	YES	NO(2)	NO	1171
(191, 75)	14	(4, 1)	3	1	YES	YES	YES	–	1172
(193, 53)	12	(4, 1)	3	1	YES	YES	YES	–	1173
(208, 37)	13	(41, 7)	11	1	YES	YES	YES	NO	1174

FIGURE 10. 46 examples for $K^2 = 4$, there are much more!

3. Deformations

A good way to get into deformations of varieties is the book by Hartshorne [H7], and the papers [S1,S3,S2], [P1], and [BW]. I will restrict the references to these papers. For $\mathbb{Q}$-Gorenstein smoothings and deformations in the KSBA moduli of surfaces, our main references are [H2] and [H5].

3.1. **General basic theory for affine and proper varieties.**

A deformation of a scheme X over Y is a flat morphism $X \to Y$ [H6, III.9]. Locally speaking, if A is a commutative ring and M is an A-module, then M is *flat* if for any exact sequence of A-modules $0 \to N \to P$ we have $0 \to N \otimes M \to P \otimes M$ is exact. For example, consider a field k and the ring inclusion $A = k[x] \to k[x,y]/(xy) = M$, and so M can be considered as A-module. Then $(x) \otimes M \to A \otimes M$ is not injective, and so it is not flat. Geometrically, this means that the projection of $\{xy = 0\}$ into the x-axis is not a deformation. The same happens for the blow-up at a nonsingular point.

By definition, let $f\colon X \to Y$ be a morphism of schemes, and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. We say that $\mathcal{F}$ is *flat* over Y at $x \in X$ if $\mathcal{F}_x$ is a flat $\mathcal{O}_{Y,y}$-module, where $f(x) = y$ and we are using the natural map $f^{\#}\colon \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$. We say that X is flat over Y if $\mathcal{O}_X$ is flat for all $x \in X$. In that case f is called a *deformation* (or a flat family, sometimes even just a family). In the following, we state some few properties from [H6, III.9]. We restrict ourselves to varieties. Deformations preserve the dimension of the fibers (and so blow-ups are not deformations). In the case of projective morphisms, flatness is the same as constant Hilbert polynomial for the fibers, in particular, the arithmetic genus is constant. Here is a useful theorem for us (see [H6, Proposition 9.7]): If $f\colon X \to Y$ is a morphisms between varieties (both reduced and irreducible), and Y is nonsingular of dimension 1, then f is flat. Hence any fibration of a surface over a nonsingular curve is flat. Let $f \in \mathbb{C}[x_1, \ldots, x_m]$ irreducible, then the projection

$$\{f = t\} \subset \mathbb{C}^m \times \mathbb{C} \to \mathbb{C}$$

is a deformation. We can write down many examples. Below we have larger families (with respect to the base) of deformations of all the ADE singularities:

$$\begin{aligned}
&A_n\ (n \geq 1) : z^2 + x^2 + y^{n+1} + t_1 y^{n-1} + \ldots + t_{n-1} y + t_n = 0\\
&D_n\ (n \geq 4) : z^2 + y(x^2 + y^{n-2}) + t_1 y^{n-2} + \ldots + t_{n-1} + t_n x = 0\\
&E_6 : z^2 + x^3 + y^4 + t_1 y^2 + t_2 y + t_3 + x(t_4 y^4 + t_5 y + t_6) = 0\\
&E_7 : z^2 + x(x^2 + y^3) + x(t_1 y + t_2) + t_3 y^4 + \ldots + t_6 y + t_7 = 0\\
&E_8 : z^2 + x^3 + y^5 + x(t_1 y^3 + \ldots + t_4) + t_5 y^3 + \ldots + t_8 = 0
\end{aligned}$$

They actually represent all possible deformations of Du Val singularities. A development of deformation theory is not possible here, so we will take the path directly to some key points, to make sense of the deformation space of a surface and computations.

Fix a complex normal variety X. We follow [BW], see much more in [O1]. An *infinitesimal deformation* of X is the existence of a commutative diagram

$$\begin{array}{ccc}
X & \longrightarrow & X' \\
\downarrow & & \downarrow \\
\operatorname{Spec}(\mathbb{C}) & \longrightarrow & \operatorname{Spec}(A)
\end{array}$$

where A is an Artin local $\mathbb{C}$-algebra (thus $\mathrm{Spec}(A)$ is one point), the map $X' \to \mathrm{Spec}(A)$ is flat, and

$$X \simeq \mathrm{Spec}(\mathbb{C}) \times_{\mathrm{Spec}(A)} X'.$$

Let $D(A)$ be the set of deformation classes of X. In particular $D(\mathbb{C}[t]/(t^2))$ are the first order deformations of X, and "represents" the tangent space $\mathbb{T}^1_X$ of the hypothetical deformation space of X. It turns out that this tangent space has a structure of $\mathbb{C}$-module. If X is a local isolated singularity, then $\mathbb{T}^1_X = \mathrm{Ext}^1(\Omega^1_X, \mathcal{O}_X)$ [S2, p.150] and it is finite dimensional. When X is compact, then $\mathbb{T}^1_X$ is finite dimensional too. When X is compact and nonsingular, then $\mathbb{T}^1_X = H^1(X, T_X)$. We recall that Ω^1_X is the sheaf of differentials on X [H6, II.8]. Its dual

$$\mathcal{H}om_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)$$

is the tangent sheaf T_X of X [H6, II.8]. Therefore, if X is nonsingular, then we have that Ω^1_X and T_X are both locally free of rank 2, and so

$$\mathrm{Ext}^1(\Omega^1_X, \mathcal{O}_X) \simeq H^1(X, T_X).$$

Example 3.1. If X is a nonsingular projective curve, then $\Omega^1_X = K_X$ and $T_X = -K_X$ (both represented by the canonical divisor class), and so if $g(X) > 1$ we obtain

$$h^1(X, T_X) = 3g - 3$$

by Riemann-Roch's theorem. This is the dimension of the moduli space of curves of genus g.

This is literal from [BW]. If $\mathbb{T}^1_X$ is finite dimensional (in our case we have it for isolated singularities or proper X), then we have, by the theory of Schlessinger [S1], that the functor D is *formally versal*. That is, we have a complete local $\mathbb{C}$-algebra R and a morphism

$$\phi\colon h_R = \mathrm{Hom}(R, \_) \to D$$

of functors (in the category of Artin local $\mathbb{C}$-algebras) such that

(i) $\phi(\mathbb{C}[t]/(t^2))$ is a bijection, and
(ii) ϕ is *smooth*, i.e., if $A' \to A$ is onto, then any

$$\begin{array}{ccc} \bar{X} & \longrightarrow & \bar{X}' \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & \mathrm{Spec}(A') \end{array}$$

comes from compatible images of h_R.

This means that all the deformations of X are encoded in the ring R. The functor D is said to be *universal* if in (ii) we have bijections. If $\mathbb{T}^0_X := H^0(X, T_X) = 0$, then it is universal (if X is affine, then this vector space is infinite dimensional; if X is of general type, then this is zero). Thus this group $\mathbb{T}^0_X$ is an obstruction to obtain an universal functor, and it represents the infinitesimal automorphism group of X.

For germs $(x \in \bar{X})$, we have existence of versal deformation (analytic) spaces Def$(x \in \bar{X})$ by Grauert, and so every deformation of $(x \in \bar{X})$ is encoded by Def$(x \in \bar{X})$, and they are determined at the first infinitesimal level. The tangent space at the point in Def$(x \in \bar{X})$ representing X is $\mathbb{T}^1_X$. The versal deformation spaces of ADE singularities appeared above. Versal deformation spaces can be arbitrarily singular in general.

For compact complex spaces X we do have versal deformation spaces Def(X) and a corresponding family $\mathcal{X} \to$ Def(X). Here a deformation is a flat and proper morphism.

- [K1] For X nonsingular, there exists a complex analytic set $(0 \in S)$ which is the versal deformation space of X. If $H^0(X, T_X) = 0$ and S is reduced, then it is universal. The space $(0 \in S)$ is the Kuranishi family. For more information see [C1].
- [D2], [G4], [P1] For any X, there exists a complex analytic set $(0 \in S)$ which is the versal deformation space of X.

Hence, in any case we have a versal deformation space. Hard questions are: How does it look like? How many components? reduced? dimensions?

The obstructions to "integrate" an infinitesimal deformation in $\mathbb{T}^1_X$ lie in a vector space $\mathbb{T}^2_X$. There is a general map

$$\mathbb{T}^1_X \to \mathbb{T}^2_X$$

whose vanishing describes all the actual deformations. When X is local complex analytic variety, then $\mathbb{T}^2_X$ is $\mathrm{Ext}^2(\Omega^1_X, \mathcal{O}_X)$ [S2]. When X is non-singular we have that $\mathbb{T}^2_X = H^2(X, T_X)$, and the map $\mathbb{T}^1_X \to \mathbb{T}^2_X$ is known as the Kuranishi map (see [C1, Theorem 8]). We will coon be interested in $\mathbb{Q}$-Gorenstein deformations, where we can understand "very well" the trilogy $\mathbb{T}^i_X$, i.e. the $\mathbb{T}^i_{QG,X}$. This will be explained in the next section. To finish, we give some useful things about the deformation theory of surfaces.

Let C be a (-1)-curve inside a nonsingular surface X. Then any deformation X_t of X carries a deformation C_t of C. This is not the case for other $(-m)$-curves, $m > 1$. The infinitesimal deformations of the pair (X, C) are encoded in $H^0(C, N_{C|X})$, and the obstructions are in $H^1(C, N_{C|X})$. But for a (-1)-curve both are zero, and so C is rigid and unobstructed, so any deformation X_t of X carries a deformation C_t of C. We can then blow-down the "(-1)-divisor" in the deformation, and so compatible with the (Castelnuovo) blow-down at each fiber. See [BHPVdV, IV.4] for more. On the other hand, let $\sigma\colon \tilde{X} \to X$ be the blow-up at a smooth point in X, and let E be the (-1)-curve. Then we have

$$0 \to \sigma^*(\Omega^1_X) \to \Omega^1_{\tilde{X}} \to \Omega^1_E \to 0,$$

and by $\otimes \Omega^2_X$, Serre duality and projection, we obtain

$$0 \to H^0(T_{\tilde{X}}) \to H^0(T_X) \to H^0(E, \mathcal{O}_E(1)) \to H^1(T_{\tilde{X}}) \to H^1(T_X) \to 0,$$

and $H^2(T_{\tilde{X}}) = H^2(T_X)$. Hence we have the same obstruction, and if $H^0(T_X) = 0$ then

$$0 \to \mathbb{C}^2 \to H^1(T_{\tilde{X}}) \to H^1(T_X) \to 0,$$

and so blowups add two dimensions to infinitesimal deformations. When X is of general type, then $H^0(T_X) = 0$. There is also a residue sequence

$$0 \to T_X(-\log C) \to T_X \to N_{C|X} \to 0$$

that can be used to understand deformations that preserve C.

If $H^1(X, T_X) = 0$ then X is rigid and K_X ample (see [BW]). A nonsingular projective surface with $H^1(X, T_X) = 0$ is obviously rigid as there is no $\mathbb{T}^1_X = 0$. (We may also have that its deformation space is a nonreduced point, so this is only a sufficient condition to be rigid.) A (-2)-curve produces an infinitesimal deformation that may not give any effective deformation [BW].

Rigid complex surfaces are very special. They could only be: del Pezzo surfaces (i.e. surfaces with $-K$ ample), Inoue surfaces, and minimal surfaces of general type [BC1, Theorem 1.3]. For surfaces of general type there are not many examples, and their geography is unknown. At the level of Chern slopes, if a minimal surface of general type X is rigid, then

$$c_1^2(X)/c_2(X) \in [5/7, 3].$$

This is implied from an estimate on the dimension of the deformation space using the Kuranishi map and Riemann-Roch. Hence between $1/5$ and $5/7$ we do have deformations for any surface! There is a lot more in [BC1]. Surfaces X with $c_1^2(X)/c_2(X) = 3$ are ball quotients, and so they are rigid. The first examples of rigid but not infinitesimally rigid varieties were worked out recently here [BP]. In [SU2], we have the first examples of rigid surfaces whose Chern slopes are arbitrarily close to the BMY-bound 3. There are other limit points for rigid surfaces in $[5/7, 3]$, but any density result or any further restriction is unknown. What could be a strategy to produce infinite families of rigid surfaces whose Chern slope sweeps a large part of $[5/7, 3]$?

Exercises.

(1) We saw that the deformation space for the Du Val singularity A_n is

$$\{z^2 + x^2 + y^{n+1} + t_1 y^{n-1} + \ldots + t_{n-1} y + t_n = 0\} \subset \mathbb{C}^3_{(x,y,z)} \times \mathbb{C}^n_{(t_1,\ldots,t_n)}.$$

Show that in the fibers we can find only A_m singularities. In fact, show that for any partition $d_1 + \ldots + d_s = n$ we have a fiber with $A_{d_1-1}, \ldots, A_{d_s-1}$ singularities.

(2) Show that $\mathbb{P}^2$ is rigid. Which del Pezzo surfaces are rigid?

(3) Let $\mathbb{F}_n$ be the Hirzebruch surface with a section of self-intersection $-n$. For example $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{F}_1$ is the blow-up of $\mathbb{P}^2$ at one point. Show that $\mathbb{F}_0$ and $\mathbb{F}_1$ are rigid.

(4) Show that for $n > 0$ a Hirzebruch surface can be explicitly presented as

$$\mathbb{F}_n = \{x_0^n y_1 = x_1^n y_2\} \subset \mathbb{P}^1_{[x_0,x_1]} \times \mathbb{P}^2_{[y_0,y_1,y_2]}.$$

Find the $\mathbb{P}^1$ fibration on $\mathbb{F}_n$ and the negative section. For $n > 1$, consider now the family

$$\mathcal{F}_n := \left\{x_0^n y_1 - x_1^n y_0 + (t_1 x_0^{n-1} x_1 + t_2 x_0^{n-2} x_1^2 + \cdots + t_{n-1} x_0 x_1^{n-1}) y_2 = 0\right\}$$

in $\mathbb{P}^1_{[x_0,x_1]} \times \mathbb{P}^2_{[y_0,y_1,y_2]} \times \mathbb{C}^{n-1}_{(t_1,\ldots,t_{n-1})}$. This gives all possible deformations of $\mathbb{F}_n$. Show that all fibers over $(t_1, \ldots, t_{n-1}) \in \mathbb{C}^{n-1} \setminus \{0\}$ a Hirzebruch surfaces $\mathbb{F}_m$ with $m \equiv n (\mathrm{mod}\ 2)$ and $m < n$. Can you find the negative section? Can you see its degeneration into some curves in $\mathbb{F}_n$?

(5) Take $\mathbb{P}^1 \times \mathbb{P}^1$. Consider the divisor D formed by 3 horizontal fibers and 3 vertical fibers. Let $f \colon X \to \mathbb{P}^1 \times \mathbb{P}^1$ be the triple cyclic cover branch along D. It has 9 A_2 Du Val singularities. Show that the minimal resolution S is a K3 surface. Compute that $h^0(T_S) = h^2(T_S) = 0$ and $h^1(T_S) = 20$. So we have a 20-dimensional unobstructed deformation space for S.

(6) $\star$ Find your own examples of rigid minimal surfaces of general type.

(7) $\star\star$ What is the exact region for Chern slopes of rigid surfaces in the allowed interval $[5/7, 3]$? We only know some families of examples, and some limit points. However, no dense region of rigid surfaces is known (see [SU2]).

3.2. $\mathbb{Q}$-Gorenstein deformations.

Our main references to understand $\mathbb{Q}$-Gorenstein deformations are [H1], [H2], and [H5]. This is how the Kollár–Shepherd-Barron moduli space of surfaces of general type locally looks like, and so its origins are in [KSB] (see also [K2, Section 6]). Behind this we have Mori's theory on families of surfaces.

The canonical cover of the T-singularity $\frac{1}{dn^2}(1, dna-1)$ is the cyclic quotient $\frac{1}{dn}(1, dn-1) \to \frac{1}{dn^2}(1, dna-1)$ which is explicitly explained in Figure 11. It gives a concrete model of the T-singularity, and it will be used to describe all $\mathbb{Q}$-Gorenstein deformations.

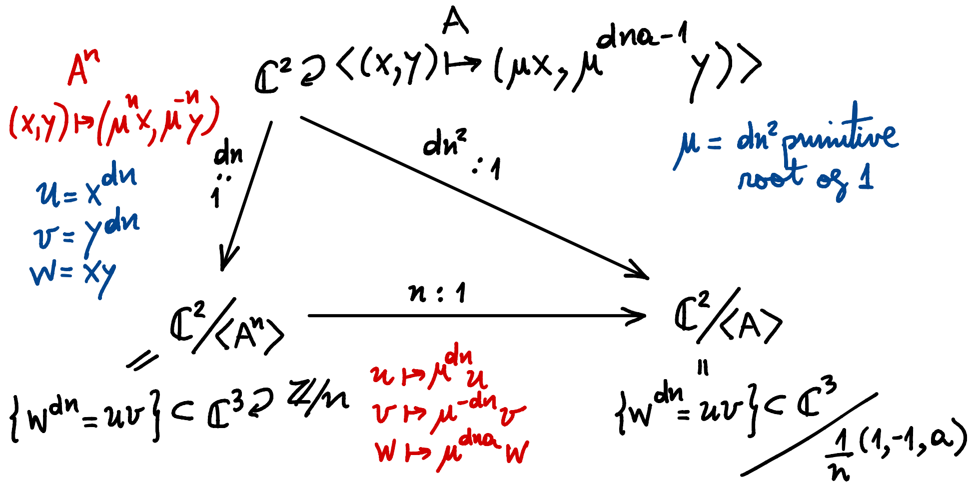


FIGURE 11. Canonical cover of a T-singularity

In fact, for any log-canonical (l.c.) singularity we can define its *canonical cover* as the cyclic cover induced by the index of the canonical class (the index of the singularity) [H2, 3.1]. It is étale in codimension 1, and it has a Gorenstein singularity.

Definition 3.2. [H2, Definition 3.1] Let $(x \in \bar{X})$ be a l.c. surface singularity, and let $(\bar{X} \subset \bar{\mathcal{X}}) \to (0 \in S)$ be a deformation of $\bar{X}$. Let n be the index of $(x \in \bar{X})$. We say that it is a $\mathbb{Q}$-*Gorenstein deformation* if it is the quotient of a $\mathbb{Z}/n$-equivariant deformation of its canonical cover.

Kollár–Shepherd-Barron [KSB] adopted another definition. When $\mathcal{X}$ is normal and S is nonsingular, this was just the condition $K_{\mathcal{X}}$ $\mathbb{Q}$-Cartier. If moreover the base is a (nonsingular) curve and the general fiber of $\mathcal{X} \to S$ is canonical, then both definitions coincide [H2, Lemma 3.4] (compare with [K2, 6.2.3]). We say that it is a $\mathbb{Q}$-*Gorenstein smoothing* if the general fiber is nonsingular. Typically our deformations will happen over a nonsingular curve germ $\mathbb{D}$, and so $\mathcal{X} \to \mathbb{D}$ is a $\mathbb{Q}$-Gorenstein smoothing if and only if all other fibers are nonsingular and $K_{\bar{\mathcal{X}}}$ is $\mathbb{Q}$-Cartier.

If W is a normal projective surface with only l.c. singularities, then a deformation $(W \subset \mathcal{W}) \to (0 \in S)$ is $\mathbb{Q}$-Gorenstein if locally induces $\mathbb{Q}$-Gorenstein deformations at every germ in W. If $\mathcal{W}$ is normal, the general fibers have only canonical singularities, and $K_{\mathcal{W}}$ is $\mathbb{Q}$-Cartier, then $\mathcal{W} \to \mathbb{D}$ is a $\mathbb{Q}$-Gorenstein deformation [H2, Lemma 3.4].

Next we have the precise picture for $\mathbb{Q}$-Gorenstein deformations of a T-singularity [HP, (2.2)] (We already reviewed the versal deformation space of ADE singularities).

Theorem 3.3. *A $\mathbb{Q}$-Gorenstein versal deformation space of the T-singularity $\frac{1}{dn^2}(1, dna-1)$, denoted by Def_{QG}, is given by*

$$\{xy = z^{dn} + t_{d-1}z^{(d-1)n} + \ldots + t_1 z^n + t_0\} \subset \frac{1}{n}(1,-1,a) \times \mathbb{C}^d,$$

via the projection $(x, y, z, t_{d-1}, \ldots, t_1, t_0) \mapsto (t_{d-1}, \ldots, t_1, t_0)$. In particular Def_{QG} has dimension d. Moreover, the possible singularities of a fiber are either $A_{e_1-1}, \ldots, A_{e_s-1}$, or $\frac{1}{e_1 n^2}(1, e_1 na-1), A_{e_2-1}, \ldots, A_{e_s-1}$, where $e_1 + \ldots + e_s = d$.

Let us fix a normal projective surface W with only T-singularities. We want to analyze the new trilogy:
$\mathbb{T}^0_{QG,W}$: Vector space of infinitesimal automorphisms of W,
$\mathbb{T}^1_{QG,W}$: Vector space of $\mathbb{Q}$-Gorenstein first order deformations of W, and
$\mathbb{T}^2_{QG,W}$: Vector space of obstructions for $\mathbb{Q}$-Gorenstein deformations.
We want to understand the versal $\mathbb{Q}$-Gorenstein deformation space $\mathrm{Def}_{QG}(W)$ of W; See [H5, Section 3].

First, it is a general theorem for automorphisms groups of proper schemes over a field k that $\mathrm{Aut}(W)$ is a group scheme locally of finite type over k. Moreover, its tangent space at the identity is $H^0(W, T_W)$ (see for example [D1, Prop.2.4]). As we work over $k = \mathbb{C}$, $\mathrm{Aut}(W)$ is reduced. Therefore, if $\mathrm{Aut}(W)$ is finite, then $H^0(W, T_W) = 0$. But indeed $\mathrm{Aut}(W)$ is finite when K_W is big by a well-known theorem of Iitaka [I, Section 11.1 for definitions, Theorem 11.12 for the result]. On the other hand, it is shown in [H1] (see [H2, Lemma 3.8]) that

$$\mathbb{T}^0_{QG,W} = H^0(W, T_W)$$

as we have an isomorphism of sheaves $T_W \simeq \mathcal{T}^0_{QG,W}$. (This is true for other singularities, not only T-singularities.)

For the analysis of the other two vector spaces, we first consider the minimal resolution of singularities $\pi \colon X \to W$. We have $\pi_* T_X = T_W$ (see [BW, Proposition 1.2]). By the Leray spectral sequence in low degree terms, we have the exact sequence (see [NSW, Lemma (2.1.3)], and [H5, Section 3])

$$0 \to H^1(W, T_W) \to \mathbb{T}^1_{QG,W} \to H^0(W, \mathcal{T}^1_{QG,W}) \to H^2(W, T_W) \to \mathbb{T}^2_{QG,W} \to 0.$$

This happens because $\mathcal{T}^i_{QG,W}$ is supported on the isolated singularities of W for $i > 0$, and $\mathcal{T}^2_{QG,W} = 0$ as the local canonical covers of W are complete intersections (see [H2, p.227]). The general set-up is described in [H5, Section 3]. The vector space $H^0(W, \mathcal{T}^1_{QG,W})$ is a direct sum of the tangent spaces of the $\mathbb{Q}$-Gorenstein deformations of each T-singularity. We have that if $\mathbb{T}^2_{QG,W} = 0$ then $\mathrm{Def}_{QG}(W)$ is nonsingular (for example, this happens when $H^2(W, T_W) = 0$). The vector space $H^1(W, T_W)$ parametrizes equisingular deformations of W.

If K_W is big, then by Riemann-Roch we can compute

$$\dim_{\mathbb{C}} \mathbb{T}^1_{QG,W} = 10\chi(\mathcal{O}_W) - 2K_W^2 + \dim_{\mathbb{C}} \mathbb{T}^2_{QG,W}, \tag{3.1}$$

and so

$$h^2(T_W) = h^0(\mathcal{T}^1_{QG}) + h^1(T_W) + 2K_W^2 - 10\chi(\mathcal{O}_W), \tag{3.2}$$

where $h^0(\mathcal{T}^1_{QG}) = \sum_i d_i$ and d_i is the d corresponding to each T-singularity in W.

If $\dim_{\mathbb{C}} H^2(W, T_W) = 0$, then $\mathrm{Def}_{QG}(W)$ is nonsingular and any local deformations of the singularities in W may be glued to obtain a global deformation of W. We say in this case that there are no *local-to-global* obstructions to deform W.

Remark 3.4. In positive characteristic we do not have necessarily the construction of $\mathrm{Def}_{QG}(W)$, and so arguments for the existence of global $\mathbb{Q}$-Gorenstein smoothings require $p_g = 0$, as they need to apply Grothendieck's effectiveness of the associated formal construction (and the Artin's algebraization of formal moduli). It is not enough to have just $H^2(T_W) = 0$ (in principle). See more details in [LN]. For example, would it be possible to prove existence of the surfaces in [RU2] in positive characteristic?

We end with an application due to the pioneering work of Lee and Park [LP1]. (This is after the construction of several exotic 4-manifolds using the rational blowdown technique of Fintushel-Stern [FS], [P2]. The point is: that construction is the diffeo analogue of $\mathbb{Q}$-Gorenstein smoothing [SSW].) They prove existence of a family of (nonsingular, minimal, projective) surfaces of general type with $p_g = 0$, $K^2 = 2$, and $\pi_1 = \{1\}$. This was the first example of a simply-connected Campedelli surface.

Lee and Park start with a particular pencil of plane cubics: Take a nonsingular conic A, and line B, and another line L. Consider the pencil $\{\nu AB + \mu L^3 = 0\}$. After blowing up 9 times, we obtain a relatively minimal elliptic fibration $Y \to \mathbb{P}^1$, whose singular fibers are of Kodaira type $IV^*, 2I_1, I_2$ (see Figure 12). In these fibrations we have a Mordell-Weil group of rank 1 with no torsion, but as shown in Figure 12 we will use only 3 sections in the construction. (In [P5], Persson produces the list of all possible configurations of singular fibers in rational elliptic fibrations with sections, and the corresponding rank and torsion of the Mordell-Weil group.)

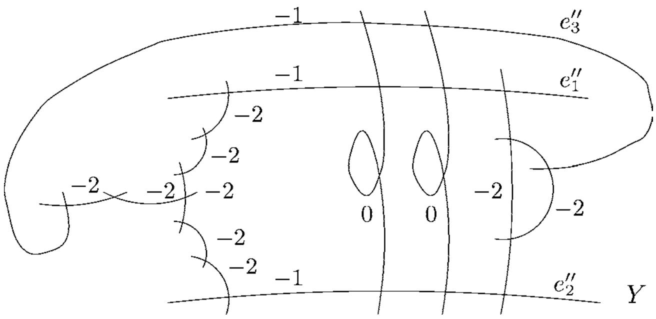


FIGURE 12. Elliptic fibration used in the first $K^2 = 2$ example of [LP1].

Consider the blow-up $X \to Y$ indicated in Figure 13 (Lee-Park [LP1] $\tilde{Z}$ is our X). We note that it is the composition of 18 consecutive blow-ups at particular points over the configuration (including infinitely near points of course). Hence $K_X^2 = -18$. We also note that X contains five Wahl chains: $[2, 10, 2, 2, 2, 2, 2, 3]$, $[2, 7, 2, 2, 3]$, $[7, 2, 2, 2]$, $[5, 2]$, and $[4]$. Let W be the contraction of these five chains. We have by Corollary 2.19

$$K_W^2 = -18 + 8 + 5 + 4 + 2 + 1 = 2.$$

Moreover we can prove that K_W is big and nef.

If we can prove that W has $\mathbb{Q}$-Gorenstein smoothings, then we would have families of surfaces of general type with $K^2 = 2$, $p_g = q = 0$. Using the Seifert-Van Kampen theorem, one can show that they all would also be simply-connected. To show the

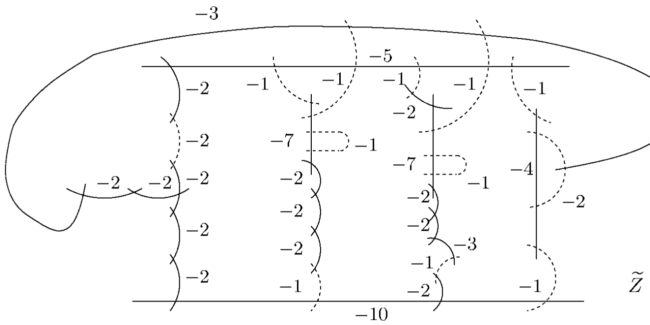


FIGURE 13. The suitable blow-ups Figure 12.

existence of $\mathbb{Q}$-Gorenstein smoothings the strategy is to prove that there are no local-to-global obstructions via the computation: $H^2(T_W) = 0$.

The following is a general strategy. As in [LP1], we have $R^i\pi_*T_X(-\log E) = 0$ for $i = 1, 2$ (and $\pi_*T_X = T_W$), where E is the exceptional (reduced) divisor of π.As we have that K_W is big, then $H^0(T_W) = 0$. Hence, by the residue sequence

$$0 \to T_X(-\log E) \to T_X \to \oplus_{C\in E} N_{C|X} \to 0,$$

we have $H^0(T_X(-\log E)) = H^0(T_X) \simeq H^0(T_W) = 0$. We also have

$$H^2(T_X(-\log E)) \simeq H^2(T_W),$$

and by Serre's duality we have $H^0(\Omega^1_X(\log E) \otimes \Omega^2_X) \simeq H^2(T_X(-\log E))$.

Let $S \to \mathbb{P}^1$ be a relatively minimal rational elliptic fibration with sections. Then $K_S \sim -F$, where F is the class of a fiber. Let F_1, F_2 two fibers of Kodaira type I_m for some ms. Then one can prove that (see for example the beginning of [PSU1, proof of Theorem 2.1])

$$H^2(T_S(-\log(F_1 + F_2))) = 0. \tag{3.3}$$

After that we can use the following two principles to add and erase curves keeping obstruction $H^2(T(-\log D))$.

(-1) We can add or erase (-1)-curves which are transversal (SNC) to the original configuration. This includes adding the (-1)-curve from the blow-up at a node of the original configuration (see for example [PSU1, Proposition 4.2 and 4.3]).

(-2) We can add or erase ADE configurations of (-2)-curves that are disjoint to the original configuration [PSU1, Theorem 4.4].

Let us come back to our example. Starting with $Y \to \mathbb{P}^1$ and F_1 and F_2 (its two nodal I_1 fibers), we obtain $H^2(T_X(-\log E)) = 0$ by applying these principles several times at the consecutive blow-ups. Therefore, $\mathbb{T}^2_{QG,W} = 0$, and so $\mathrm{Def}_{QG}(W)$ is smooth of dimension $10 - 2K^2 = 6$, and each of the 5 singularities contribute with one dimension. We have that $h^1(T_W) = 1$.

Remark 3.5. I think one can degenerate further by just degenerating the conic into two lines. This would explain the extra dimension from $h^1(T_W) = 1$. What degeneration do we get as a KSBA surface? Do we acquire an extra Wahl singularity?

Later Park, Park, Shin were able to find the analogue surfaces for $K^2 = 3$ [PPS1], and $K^2 = 4$ [PPS2]. For this last case, these examples are the only available surfaces in the literature (two examples), in reality there are hundreds of them (see 46 in Table 10).

Note that a restriction to use this method is

$$10\chi - 2K^2 > 0,$$

and so for $p_g = q = 0$ we can do it only for $K^2 = 1, 2, 3, 4$. For more $p_g = 0$ examples in general see for example [SU1, RU3], where we also use other singularities.

Exercises.

(1) Verify Equations 3.1 and 3.2 (see [RU3, Theorem 4.1]).

(2) Show that for any $d \geq 1$ there is a $d\colon 1$ cover of the Wahl singularity $\frac{1}{n^2}(na-1)$ given by $\frac{1}{dn^2}(1, dna'-1) \xrightarrow{d:1} \frac{1}{n^2}(1, na-1)$, where a' satisfies that $a \equiv da' (\text{mod } n)$, which fits in the following commutative diagram

$$\begin{array}{ccc} \mathbb{C}^2 & \xrightarrow{(x,y)\mapsto(x^d,y^d)} & \mathbb{C}^2 \\ \downarrow & & \downarrow \\ \frac{1}{dn^2}(1, dna'-1) & \xrightarrow{d:1} & \frac{1}{n^2}(1, na-1) \end{array}$$

Are there induced smoothings?

(3) Find equations for the pencil and singular fibers of the example in Figure 12. Show that K_W is big and nef for the W constructed from Figure 13.

(4) Show that Equation 3.3 holds. Verify that we have no local-to-global obstructions to deform W constructed from Figure 13, using the add and erase principles. In fact, you can show that $\pi = \{1\}$ for the general fiber, using Van Kampen Theorem, check [LP1].

(5) On the other hand, if we start with 3 or more complete singular fibers in an elliptic fibration with sections, then we will find out positive obstruction. Show one example where $H^2(T_W) \neq 0$ (see [RU3, Section 4]).

(6) Enriques surfaces degenerate to surfaces W with T-singularities of type $\frac{1}{4d}(1, 2d-1)$ only. There are many ways to do it, in [U2] you can have many. In indicated in [U2, Section 6], they can be used to understand families of $\mathbb{Q}$-Homology projective planes with numerically trivial canonical class. Realize at least one of these one-dimensional families as a $\mathbb{Q}$-Gorenstein smoothing of some rational surface.

(7) $\star$ Sometimes, even though there are obstructions, one can construct anyway a complex smoothing. For example, in the presence of some symmetry, whose quotient has no obstructions, for details see [LP2, Theorem 3]. That is applied to a well-known family of Horikawa surfaces. Find your own application for some other surfaces of general type.

(8) $\star\star$ Construct a complex simply-connected $p_g = 0$ surface of general type with $K^2 \geq 5$. With the strategy of $\mathbb{Q}$-Gorenstein smoothings, you can start with the examples in [RU3] for $K^2 = 5$.

3.3. **Kollár–Shepherd-Barron correspondence.**

For this section, the fundamental source is [KSB, Section 3]. The main point in that section was to prove that the components of the reduced versal deformation space of a c.q.s. can be described in a one-to-one correspondence via "P-resolutions", and that each deformation in a component can be obtained using $\mathbb{Q}$-Gorenstein deformations of the associated P-resolution. Behind this we have Mori theory of a family of surfaces over a nonsingular curve, which is explained at the very beginning of the paper [KSB, Theorem p.301].

Let $0 < q < m$ be coprime integers. Let us consider the c.q.s. $\overline{W} := \frac{1}{m}(1,q)$. As explained in the previous sections, there exists a versal deformation space $\mathrm{Def}(\overline{W})$. The infinitesimal deformations were described here [R2]. Equations for $\mathrm{Def}(\overline{W})$ were investigated here [S8].

Definition 3.6. [KSB, Definition 3.8] A *P-resolution* of $\overline{W}$ is a partial resolution $f\colon W \to \overline{W}$ such that W has only T-singularities, and K_W is ample relative to f.

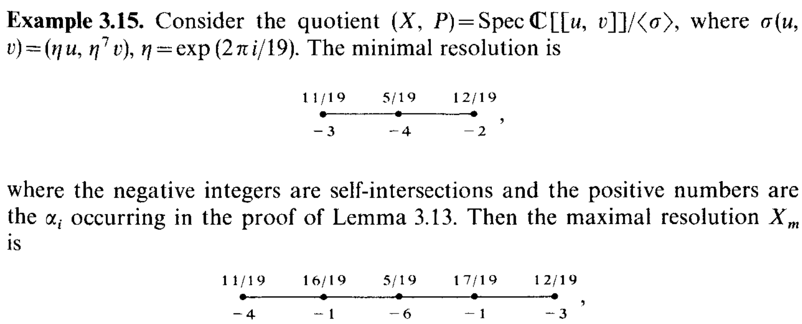
P-resolutions were actually defined for any quotient singularity, and for the Kollár's conjecture (see Remark 3.10) we would like to define it for any rational singularity, where T-singularities may not be the only participants anymore.

Definition 3.7. [KSB, Definition 3.12] A resolution $f\colon X \to \overline{W}$ is *maximal* if $K_X \equiv f^*(K_{\overline{W}}) - \sum_i a_i E_i$, where $0 < a_i < 1$, and for any proper birational morphism $g\colon Z \to X$ that it is not isomorphism, we have $K_Z \equiv h^*(K_W) - \sum_j b_j F_j$, where $h = f \circ g$ and some $b_j \leq 0$.

Lemma 3.8. [KSB, Lemma 3.13 and 3.14] *Any $\overline{W}$ admits a unique maximal resolution, and it dominates any P-resolution of $\overline{W}$.*

Proof. They are short proofs in [KSB]. Essentially, it uses what we know about discrepancies and negative definiteness of matrices of exceptional divisors. □

Thus, in particular, for a P-resolution we obtain that the exceptional divisor is a chain of $\mathbb{P}^1$s and T-singularities are at the nodes in this chain.


Example 3.15. Consider the quotient $(X, P) = \operatorname{Spec} \mathbb{C}[[u, v]]/\langle\sigma\rangle$, where $\sigma(u, v) = (\eta u, \eta^7 v)$, $\eta = \exp(2\pi i/19)$. The minimal resolution is

where the negative integers are self-intersections and the positive numbers are the α_i occurring in the proof of Lemma 3.13. Then the maximal resolution X_m is

FIGURE 14. An example from [KSB].

The discrepancies in the example Figure 14 are $-\frac{11}{19}, -\frac{14}{19}, -\frac{7}{19}$ from left to right. The discrepancies in the maximal resolution are $-\frac{11}{19}, -\frac{6}{19}, -\frac{1}{19}, -\frac{14}{19}, -\frac{2}{19}, -\frac{7}{19}$. From here one can check the existence of exactly 3 P-resolutions (as correctly stated in [KSB, Example 3.15]):

1. Minimal resolution with the (-2)-curve contracted.
2. Minimal resolution with the (-4)-curve contracted.
3. $[4] - (1) - [5,2]$ which means contraction of $[4]$ and $[5,4]$ from the blow-up at the intersection between the (-3)-curve and the (-4)-curve in the minimal resolution.

Theorem 3.9. [KSB, Theorem 3.9] *There is a one-to-one correspondence between the irreducible components of* $\mathrm{Def}(\overline{W})$ *and the P-resolutions of* $\overline{W}$*. In addition, the* $\mathbb{Q}$*-Gorenstein deformations of a P-resolution maps onto the corresponding component of* $\mathrm{Def}(\overline{W})$*.*

Remark 3.10. Kollár conjectured a bigger picture for deformations of rational singularities. In [K2, Section 3] he suggested that any deformation of a two-dimensional rational singularity should be the blow-down deformation of a $\mathbb{Q}$-Gorenstein deformation of some P-resolution (where, of course, the singularities that admit this type of deformation may no longer be T-singularities). Stevens developed ideas around this conjecture in [S7, Section 14]. Some recent preprints on this conjecture for sandwiched singularities are [PS1, JS], which have a lot to do with the birational geometry in this text.

How to find these P-resolutions in a more systematic way? Is it possible to count the irreducible components of $\mathrm{Def}(\overline{W})$? Which are their dimensions?

Here is when Christophersen [C2] and Stevens [S9] enter to the picture, and so the zero continued fractions in Section 1.3. We recall that these continued fractions admit entries which are greater than or equal to 1, without 1s it is not possible to get zero!

Let us consider the Hirzebruch-Jung continued fraction associated to $\frac{1}{m}(1,q) = \overline{W}$

$$\frac{m}{q} = e_1 - \cfrac{1}{e_2 - \cfrac{1}{\ddots - \frac{1}{e_\ell}}},$$

and its dual

$$\frac{m}{m-q} = b_1 - \cfrac{1}{b_2 - \cfrac{1}{\ddots - \frac{1}{b_s}}}.$$

We define the set of zero continued fractions

$$K(\overline{W}) = \{[k_1, \ldots, k_s] = 0\colon \text{ such that } 1 \leq k_i \leq b_i\}.$$

It turns out that there is a bijection between this set and the set of P-resolutions of $\overline{W}$. How? In [PPSU] it is explained a geometric way to do this, whose numerical part is the following. Before that, we describe another one-to-one correspondence but now with M-resolutions [BC2].

Definition 3.11. An *M-resolution* of $\overline{W}$ is a partial resolution $f\colon W \to \overline{W}$ such that W has only Wahl singularities, and K_W is nef relative to f.

At the end of the day, given a P-resolution, we can obtain an M-resolution by replacing the singularities of type A by their minimal resolutions, and by replacing a T-singularity $\frac{1}{dn^2}(1, dna-1)$ by a chain of $d-1$ $\mathbb{P}^1$s with Wahl singularities $\frac{1}{n^2}(1, na-1)$ at the nodes. This was explained before in the section on T-singularities. In this way, we obtain a one-to-one correspondence between M-resolutions and P-resolutions. Again, all deformations of $\overline{W}$ come from blowing down $\mathbb{Q}$-Gorenstein deformations of an M-resolution W [BC2].

Notation 3.12. Let $[\binom{n}{a}]$ denote the Hirzebruch–Jung continued fraction of $\frac{n^2}{na-1}$. Calculations with Wahl surfaces will use this notation to indicate self-intersections in the chain of rational curves in the minimal resolution of the Wahl surface. For example, our fixed M-resolution $W^+ \to \overline{W}$ is represented by

$$\left[\begin{pmatrix} n_0 \\ a_0 \end{pmatrix}\right] - (c_1) - \left[\begin{pmatrix} n_1 \\ a_1 \end{pmatrix}\right] - (c_2) - \ldots - (c_r) - \left[\begin{pmatrix} n_r \\ a_r \end{pmatrix}\right] \quad \longrightarrow \quad [e_1, \ldots, e_\ell].$$

Here $[\binom{n_i}{a_i}]$ represents the Wahl singularity $p_i \in W^+$ for $i = 0, \ldots, r$, and (c_i) represents the rational curve $\Gamma_i \subset W^+$ for $i = 1, \ldots, r$, so that its proper transform in the minimal resolution of W^+ has self-intersection $-c_i$. The arrow $\longrightarrow$ means that the chain contracts to $[e_1, \ldots, e_\ell]$ by consecutively contracting (-1)-curves.

Remark 3.13. We have

$$[b_s, \ldots, b_1] - (1) - \left[\begin{pmatrix} n_0 \\ a_0 \end{pmatrix}\right] - (c_1) - \left[\begin{pmatrix} n_1 \\ a_1 \end{pmatrix}\right] - (c_2) - \ldots - (c_r) - \left[\begin{pmatrix} n_r \\ a_r \end{pmatrix}\right] = 0.$$

We use the geometric procedure in [PPSU, Cor.10.1], which interprets the zero continued fraction of the Wahl resolution as follows:

Algorithm 1 (for M-resolutions)**.**

(0) If $i_1 = 1$, then $n_0 = a_0 = 1$. Otherwise

$$\frac{n_0}{n_0 - a_0} = [b_1, \ldots, b_{i_1 - 1}].$$

(1) At the beginning of the M-resolution (see Notation 3.12) we have d_{i_1} Wahl chains $[\binom{n_0}{a_0}]$ as follows:

$$\underbrace{\left[\begin{pmatrix} n_0 \\ a_0 \end{pmatrix}\right] - (1) - \left[\begin{pmatrix} n_0 \\ a_0 \end{pmatrix}\right] - (1) - \ldots - (1) - \left[\begin{pmatrix} n_0 \\ a_0 \end{pmatrix}\right]}_{d_{i_1}} - (c_{d_{i_1}}) - \ldots.$$

We can blow-down the indicated (-1)-curves in the chain

$$[b_s, \ldots, b_1] - (1) - \underbrace{\left[\begin{pmatrix} n_0 \\ a_0 \end{pmatrix}\right] - (1) - \ldots - (1) - \left[\begin{pmatrix} n_0 \\ a_0 \end{pmatrix}\right]}_{d_{i_1}} - (c_{d_{i_1}}) - \ldots.$$

and all subsequently appearing (-1)-curves consecutively until we obtain the new chain with the following curves:

$$[b_s, \ldots, b_{i_1+1}, b_{i_1} - d_{i_1}, b_{i_1 - 1}, \ldots, b_1] - (c_{d_{i_1}}) - \left[\begin{pmatrix} n_{d_{i_1}} \\ a_{d_{i_1}} \end{pmatrix}\right] - \ldots - (c_r) - \left[\begin{pmatrix} n_r \\ a_r \end{pmatrix}\right].$$

(2) If $b_{i_1} - d_{i_1} = 1$, then we contract this (-1)-curve and all new (-1)-curves in the subchain $[b_s, \ldots, b_{i_1+1}, b_{i_1} - d_{i_1}, b_{i_1-1}, \ldots, b_1]$ until there are none.

(3) Then the original $(-c_{d_{i_1}})$-curve becomes a (-1)-curve, and we have

$$\frac{n_{d_{i_1}}}{n_{d_{i_1}} - a_{d_{i_1}}} = [b_1, \ldots, b_{i_1 - 1}, b_{i_1} - d_{i_1}, b_{i_1+1}, \ldots, b_{i_2 - 1}],$$

if this is not 1. Otherwise, $n_{d_{i_1}} = a_{d_{i_1}} = 1$.

(4) We now repeat starting in (1) with the d_{i_2}.

(5) We end with $[k_s, \ldots, k_1] = [\ldots, b_{i_e} - d_{i_e}, \ldots, b_{i_1} - d_{i_1}, \ldots] = 0$, which is the zero continued fraction corresponding to the M-resolution.

Example 3.14. Take $\frac{m}{q} = \frac{89}{33} = [3,4,2,2,4]$. Take the M-resolution $W \to \overline{W}$ given by

$$\left[\binom{2}{1}\right] - (1) - \left[\binom{3}{1}\right] - (2) - \left[\binom{2}{1}\right] = [4] - (1) - [5,2] - (2) - [4] \quad \to \quad [3,4,2,2,4].$$

We have $\frac{m}{m-q} = [2,3,2,5,2,2]$, and so $[2,2,5,2,3,2,1,3,4,2,2,4] = 0$. The element in $K(\overline{W})$ corresponding to W is $[2,2,1,5,1,2] = 0$. Thus $d_1 = 0$, $d_2 = 1$, $d_3 = 1$, $d_4 = 0$, $d_5 = 1$, and $d_6 = 0$.

Note that [KSB, Example 3.15] admits three M-resolutions:

$$(3) - (4) - (2), \qquad \left[\binom{2}{1}\right] - (1) - \left[\binom{3}{1}\right], \qquad (3) - \left[\binom{2}{1}\right] - (2).$$

As explained at the end of [KSB, Section 3], the dimension of the $\mathbb{Q}$-Gorenstein deformation space of a P-resolution $W \to \overline{W}$ can be computed explicitly. As these $\mathbb{Q}$-Gorenstein deformations are in bijection with the corresponding component of $\mathrm{Def}(\overline{W})$, we obtain the formula

$$\sum_{i=1}^{\ell}(e_i - 3) + 2\sum_{i=1}^{s} b_i - 2\sum_{i=1}^{s} k_i - 2,$$

for the dimension of the component corresponding to the zero continued fraction in $K(\overline{W})$ given by $[k_1, \ldots, k_s]$. In particular, the difference between dimensions of any two components corresponding to $\underline{k}, \underline{k}'$ is

$$2\left|\sum_{i=1}^{s} k_i' - \sum_{i=1}^{s} k_i\right|.$$

The zero continued fraction corresponding to the minimal resolution is

$$[1, 2, \ldots, 2, 1],$$

and gives the component of bigger dimension, this is the Artin component (deformations that admit a simultaneous resolution of singularities after a finite base change).

Exercises.

(1) Find all M-resolutions corresponding to the c.q.s. whose HJ continued fraction is

$$[\underbrace{2,\ldots,2}_{a}, 3, \underbrace{2,\ldots,2}_{b}, 3, \underbrace{2,\ldots,2}_{c}].$$

(2) What is the maximal and the minimal dimension of a component of the deformation space of a c.q.s.?
(3) $\star$ Find a characterization of c.q.s. whose deformation space is irreducible.
(4) Find a characterization of c.q.s. whose deformation space has $\frac{1}{s}\binom{2(s-1)}{s-1}$ where s is the length of its dual fraction. This is a lot of components!
(5) For each P-resolution of a c.q.s. compute the Milnor number of a general smoothing in the corresponding component.

Using the computer program MNres [Z1] one can find all M-resolutions (and N-resolutions), which will be introduced later) of the c.q.s. $\frac{1}{85}(1,49)$. We have

$$\frac{85}{49} = [2,4,5,2,2],$$

and $\frac{85}{36} = [3,2,3,2,2,4]$. This c.q.s. has a deformation space with 5 irreducible components. For each of them, we list the corresponding: zero continued fraction, dimension of the component and the M-resolution.

$[1,2,2,2,2,1]$, dimension is 10, $(2)-(4)-(5)-(2)-(2)$ (minimal resolution)
$[2,1,3,2,2,1]$, dimension is 8, $(2)-[\binom{2}{1}]-(5)-(2)-(2)$
$[1,2,3,2,1,3]$, dimension is 6, $(2)-(4)-[\binom{3}{1}]-(2)$
$[2,2,3,1,2,4]$, dimension is 2, $(2)-[\binom{7}{2}]$ (extremal P-resolution)
$[3,1,3,2,1,4]$, dimension is 2, $[\binom{3}{2}]-(1)-[\binom{4}{1}]$ (extremal P-resolution)

This is a c.q.s. that admits 2-zero continued fractions (see Theorem 1.21). These are extremal P-resolutions, which are main protagonists in the next section.

4. W-SURFACES

4.1. **Picard group, class group, and topology again.**

Arbitrary degenerations of nonsingular projective surfaces over a curve germ $\mathbb{D}$ into a surface with only c.q.s can always be reduced to a $\mathbb{Q}$-Gorenstein smoothing of a surface W with only Wahl singularities (see [HTU, Section 5]). In [K2, Theorem 3.4.2] it is stated the following result due to various authors, among them Tsunoda, Mori, Kawamata: Any 1-parameter degeneration of non-ruled surfaces can be modified into a minimal model family (canonical class relatively nef) whose central fiber has only

- Wahl singularities,
- simple normal crossings singularities ($\{xy=0\}$ or $\{xyz=0\}$ in $\mathbb{C}^3$), and/or
- orbifold double normal crossings singularities [H5, Section 5].

That shows the relevance of this type of degenerations. Next we collect the information on a normal projective surface W with only Wahl singularities and a $\mathbb{Q}$-Gorenstein smoothing of W over $\mathbb{D}$ [U4, Section 2].

Definition 4.1. A *W-surface* is a normal projective surface W together with a proper deformation $(W \subset \mathcal{W}) \to (0 \in \mathbb{D})$ such that

(1) W has at most Wahl singularities;
(2) $\mathcal{W}$ is a normal complex 3-fold with $K_{\mathcal{W}}$ $\mathbb{Q}$-Cartier;
(3) the fiber W_0 is reduced and isomorphic to W;
(4) the fiber W_t is nonsingular for $t \neq 0$.

The W-surface is said to be *smooth* if W is nonsingular.

For a W-surface the invariants $q(W_t)$, $p_g(W_t)$, $K^2_{W_t}$, $\chi_{\text{top}}(W_t)$ remain constant for every $t \in \mathbb{D}$. The fundamental group of W_0 and W_t may differ (e.g. Enriques surfaces). A W-surface is *minimal* if K_W is nef, and so K_{W_t} is nef for all t [U4]. If a W-surface is not minimal, then we can run explicitly the MMP relative to $\mathbb{D}$, which is fully worked out in [HTU]. It arrives at a minimal model or other outcomes, they are explained in [U4, Section 2]. (This MMP will be elaborated in the next sections.) When K_W is nef and big, the canonical model of $(W \subset \mathcal{W}) \to (0 \in \mathbb{D})$ has only T-singularities (this means Du Val and $\frac{1}{dn^2}(1, dna-1)$). (Cf. [U4, Section 2] and [U3, Sections 2 and 3].)

We now go step by step from local to global. We mainly follow [W2] and [LW] for Milnor fibers and local topological facts. Let us first consider a germ $(P \in W)$ of a normal surface singularity. A smoothing of W comes with a *Milnor fiber* M, which is just the diffeomorphism type of the smooth fiber W_t. It is a real compact 4-manifold with boundary L, which turns out to be the *link of the singularity*: If $(P \in W) \subset \mathbb{C}^N$, then the link of $(P \in W)$ is the intersection of the boundary of a ball around P with W, and so it is a 3-dimensional manifold. The Milnor fiber has the homotopy type of a finite CW-complex of dimension 2. In particular, it has no homology after dimension 2. For the Betti numbers

$$b_i(M) = \text{rank}(H_i(M))$$

we have:

- Wahl conjectured in [W2] that $b_1(M) = 0$ for any singularity. This was proved in [GS]. One can easily check this for rational singularities, since in this case $b_1(L) = 0$ (Mumford's computation [M5, p.235]), and one has $b_1(M) \leq b_1(L)$.[6]

[6]There is some discussion on $\pi_1(M)$ in [GS, §4].

- We call $b_2(M)$ the *Milnor number* of the corresponding smoothing. This is the number of vanishing cycles in the smoothing. It depends on the smoothing of course, no just the singularity. We have that $H_2(M)$ has a bilinear symmetric form, and let μ_0, μ_-, μ_+ be the number of 0s, -1s, and $+1$s in a real diagonalization. Of course $b_2(M) = \mu_0 + \mu_+ + \mu_-$. By a theorem of Steenbrink [S5, Theorem 2.24] we have
$$\mu_0 + \mu_+ = 2p_g(P \in W).$$
Hence, μ_0, μ_+ do not depend on the smoothing, and, for example, if the smoothing has Milnor number equal to zero, then $(P \in W)$ must be a rational singularity.

For example, for singularities of type A_n, D_n, or E_n we have only one type of smoothing (i.e. one Milnor fiber) and $b_2 = n$.

This operation of smoothing $Y \rightsquigarrow W$ is a smooth surgery that can be performed in the smooth category, think for example about the rational blowdown [FS]. (It can actually be performed in complete generality in the symplectic category by [PS2].) We have a continuous map of pairs
$$f\colon (W^o, \cup L_i) \to (Y, \cup M_i),$$
where W^0 is the complement of the singularities P_i in W, L_i are the links at the P_i, and M_i are the Milnor fibers used in the smoothing $Y \rightsquigarrow W$ for each P_i. One can show that this induces a commutative diagram between long exact sequences (see [TU]):
$$\begin{array}{ccccccccccccc}
\bigoplus_i H_2(L_i) & \longrightarrow & H_2(W^o) & \longrightarrow & H_2(W) & \longrightarrow & \bigoplus_i H_1(L_i) & \longrightarrow & H_1(W^o) & \longrightarrow & H_1(W) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \| & & \downarrow & & \downarrow & & \| & & \\
\bigoplus_i H_2(M_i) & \longrightarrow & H_2(Y) & \longrightarrow & H_2(W) & \longrightarrow & \bigoplus_i H_1(M_i) & \longrightarrow & H_1(Y) & \longrightarrow & H_1(W) & \longrightarrow & 0
\end{array}$$
We note that for c.q.s. $\frac{1}{m}(1,q)$, its link is the Lens space defined as the quotient of the 3-sphere S^3 by the action $(x,y) \mapsto (\mu_m x, \mu_m^q y)$, where μ_m is primitive root of 1 and $S^3 \subset \mathbb{C}^2$. We have $H_0(L) = H_3(L) = \mathbb{Z}$, $H_1(L) = \mathbb{Z}/m$, and $H_2(L) = 0$. For a $\mathbb{Q}$-Gorenstein smoothing of a Wahl singularity $\frac{1}{n^2}(1, na-1)$ we have that the homology of the Milnor fiber is
$$H_0(M) = \mathbb{Z}, H_1(M) = \mathbb{Z}/n, \text{ and } H_2(M) = 0.$$
By [LW, Lemma 5.1] we have that (in general) the morphism $H_1(L) \to H_1(M)$ is onto.

Remark 4.2. Let $Y \rightsquigarrow W$ be any smoothing. Let μ_i be the corresponding Milnor number from each P_i induced by the deformation, and let E_i be the exceptional divisor in X over P_i, where $X \to W$ is the minimal resolution of singularities. We have for the topological Euler characteristic
$$\chi_{top}(X) = \chi_{top}(W \setminus \cup_i P_i) + \sum_i \chi_{top}(E_i) = \chi_{top}(Y) - \sum_i (1 + \mu_i) + \sum_i \chi_{top}(E_i)$$
Here we are using that $b_1(\text{Milnor fiber}) = 0$ for any normal singularity and any smoothing. On the other hand we always have $\chi(\mathcal{O}_W) = \chi(\mathcal{O}_Y)$, and so
$$K_Y^2 + \sum_i \mu_i = K_X^2 + \sum_i \big(b_2(E_i) - b_1(E_i)\big) + 12 \sum_i p_g(P_i \in W)$$
using the Noether formula. The right hand side depends only on the singularity, and the left on the smoothing.

Let us assume the situation of a W-surface and the corresponding smoothing $(W \subset \mathcal{W}) \to (0 \in \mathbb{D})$. In addition, assume $H^1(W, \mathcal{O}_W) = H^2(W, \mathcal{O}_W) = 0$. Then we have $H_2(W) = \mathrm{Cl}(W)$, $H_2(Y) = \mathrm{Cl}(Y) = \mathrm{Pic}(Y)$, and $H_1(L_i)$ is the local class group of $P_i \in W$, see for example [K1, Prop.4.2 and 4.11]. Therefore, we obtain

$$\begin{array}{ccccccccc} 0 \longrightarrow \mathrm{Pic}(W^o) & \longrightarrow & \mathrm{Cl}(W) & \longrightarrow & \bigoplus_i \mathbb{Z}/n_i^2 & \longrightarrow & H_1(W^o) & \longrightarrow & H_1(W) \longrightarrow 0 \\ \downarrow & & \| & & \downarrow & & \downarrow & & \| \\ 0 \longrightarrow \mathrm{Pic}(Y) & \longrightarrow & \mathrm{Cl}(W) & \longrightarrow & \bigoplus_i \mathbb{Z}/n_i & \longrightarrow & H_1(Y) & \longrightarrow & H_1(W) \longrightarrow 0 \end{array}$$

In particular, from this one can read the Picard group of Y as the kernel of $\mathrm{Cl}(W) \to \bigoplus_i \mathbb{Z}/n_i$. For any p_g, this group can be thought of as a generalization of the Coble-Mukai lattice (up to torsion) of an Enriques surface. The explicit description and applications will be written elsewhere. For now, we finish with the example of an Enriques surface as in [U2].

Let W be an Enriques W-surface with K_W nef, i.e., the general fiber Y is an Enriques surface. Let $\pi\colon X \to W$ be the minimal resolution of W. Kawamata proves that the monodromy for this type of degenerations is trivial [K2, Section 2] [7], and that W can have only singularities of type $\frac{1}{4}(1,1)$ [K2, Theorem 4.1] (i.e. these are the flower pot degenerations of Persson). Let $\{C_1, \ldots, C_s\}$ be the disjoint exceptional (-4)-curves of π. Then

$$-2K_X \sim C_1 + \ldots + C_s,$$

and so X is a *Coble surface of K3 type*. In this way, we have two options for X (see [KD, Prop. 9.1.4]): it is the blow-up of either

1. an Halphen surface of index two [8] over the singularities of one reduced fiber, which is of type II, III, IV, or I_n, or
2. a jacobian rational minimal elliptic fibration over the singularities of two reduced fibers of type II, III, IV, or I_n.

We will only consider the case when the singular fibers are of type I_n, the other situations are degenerations. Hence, for one singular fiber we have I_s, and for two we have I_{s_1} and I_{s_2} with $s_1 + s_2 = s$. We have that $1 \leq s \leq 10$ by [KD, Corollary 9.1.5]. One can actually prove that such a W has no local-to-global obstructions to deform, and that any $\mathbb{Q}$-Gorenstein smoothing of it is an Enriques surface (see [U3, Theorem 4.2(0)], where there is a result for more general elliptic fibrations).

Let β_i be the class of C_i in $\mathrm{Pic}(X)$. As in [KD, Chapter 9], we define the *Coble-Mukai lattice* of X as

$$\mathrm{CM}(X) := \{x \in \widetilde{\mathrm{Pic}}(X) \colon x \cdot \beta_i = 0 \text{ for all } i\}$$

where $\widetilde{\mathrm{Pic}}(X)$ is the lattice in $\mathrm{Pic}(X)_{\mathbb{Q}}$ generated by $\mathrm{Pic}(X)$ and the rational classes $\frac{1}{2}\beta_i$.

By the work above, we have the short exact sequence

$$0 \to \mathrm{Pic}(Y) \to \mathrm{Cl}(W) \to (\mathbb{Z}/2)^{s-1} \to 0.$$

[7] In fact [K2, Section 2] has various things on topology and Hodge structures for $\mathbb{Q}$-Gorenstein smoothings.

[8] An Halphen surface of index two is a rational elliptic surface with a multiplicity two fiber, and no (-1)-curves in the fibers.

(Note that for $s=1$ it also works, and so $\mathrm{Pic}(Y) \simeq \mathrm{Cl}(W)$ in that case.) From now on we refer to [U2].

Lemma 4.3. *The image of* $\mathrm{Pic}(Y)$ *in* $\mathrm{Cl}(W)$ *is the set of classes whose proper transform have even intersection number with each* C_i.

Theorem 4.4. *The image of* $\mathrm{Pic}(Y)$ *in* $\mathrm{Cl}(W)$ *quotient by* $\langle \mathcal{O}_Y(K_Y) \rangle$ *is isomorphic to* $CM(X)$.

Proof. We prove this through the pull-back morphism

$$\pi^* \colon \mathrm{Cl}(W) \to \widetilde{\mathrm{Pic}}(X).$$

First, the pull-back on any class in $\mathrm{Cl}(W)$ is orthogonal to all β_i. So we now restrict to $\pi^* \colon \mathrm{Pic}(Y) \to \mathrm{CM}(X)$. Let $D = D' + \sum_{i=1}^s \frac{a_i}{2}\beta_i$ in $\mathrm{CM}(X)$ with D' not supported at the β_i, and $a_i \in \mathbb{Z}$. Then by definition $D \cdot \beta_i = 0$ and so $D' \cdot C_i$ is even for all i, and so π^* is onto, by Lemma 4.3. If $\pi^*(D) = D' + \sum_{i=1}^s \frac{a_i}{2}\beta_i = 0$, then $\pi^*(2D) = 0$ in $\mathrm{Pic}(X)$. Say that $D \neq 0$ in $\mathrm{Pic}(Y)$, and so we have a numerical 2-torsion class, and this implies $D \sim K_Y$ by Riemann-Roch on D. □

In [KD, Theorem 9.2.15], it is proved that the Coble-Mukai lattice of a Coble surface is isomorphic to the Enriques lattice over $\mathbb{C}$ by a different method.

Exercises.

(1) As we saw, a surface singularity $(P \in \overline{W})$ admitting a smoothing with Milnor number equal to zero must be rational. Let $X \to \overline{W}$ be its minimal resolution. Show that the difference $K_Y^2 - K_X^2$ is equal to the number of exceptional curves over P. (This is a surprising formula because in principle we would need to compute discrepancies and intersections between exceptional curves.) In particular you have now a formula for all $\mathbb{Q}$HD singularities in [BS], which are conjectured to be all $\mathbb{Q}$HD singularities.

(2) Using the explicit $\mathbb{Q}$-Gorenstein smoothing in Figure 11, compute the fundamental group of the associated Milnor fiber for any T-singularity.

(3) An Enriques surface S can be seen as an elliptic fibration $S \to \mathbb{P}^1$ with $p_g = 0$ and exactly two multiplicity 2 fibers. A computation/definition for the Coble-Mukai lattice can be done for any elliptic fibration $S \to \mathbb{P}^1$ with $p_g = 0$ and exactly two multiplicity n fibers. Try an example following the details we gave for Enriques surfaces.

(4) $\star$ Let $(P \in \overline{W})$ be a c.q.s. and let $W \to \overline{W}$ be an M-resolution with r $\mathbb{P}^1$s and Wahl singularities $\frac{1}{n_i^2}(1, n_i a_i - 1)$ for $i = 0, 1, \ldots, r$. Let M be the Milnor fiber corresponding to that M-resolution. Compute $H_0(Y) \simeq \mathbb{Z}$,

$$H_1(Y) \simeq \mathbb{Z}/\mathrm{gcd}(n_0, \ldots, n_r) \quad H_2(Y) \simeq \mathbb{Z}^r.$$

4.2. **MMP for W-surfaces I.**

Semi-stable minimal model program [KM1, Section 7] is the process to obtain a minimal model for a degeneration of surfaces, this is, a birational deformation with a 3-fold with correct singularities (terminal) and nef canonical class. If this is a degeneration of surfaces of general type, then we obtain the unique family of canonically polarized surfaces from the canonical model of this minimal model. That is the whole point of the compactification proposed by Kollár–Shepherd-Barron [KSB]. Now, for an arbitrary degeneration of surfaces, we would have to resolve the 3-fold family and then apply semi-stable MMP. In the case of a W-surface we do not need to do that, we directly work with the W-surface, obtaining in the process only W-surfaces.

The following is the working object in [KM2].

Definition 4.5. An *extremal neighborhood* $(\Gamma \subset \mathcal{W}) \to (P \in \overline{\mathcal{W}})$ is a proper birational morphism between normal 3-folds $F\colon \mathcal{W} \to \overline{\mathcal{W}}$ such that

(1) The canonical class $K_{\mathcal{W}}$ is $\mathbb{Q}$-Cartier and $\mathcal{W}$ has only terminal singularities.
(2) There is a distinguished point $P \in \overline{\mathcal{W}}$ such that $F^{-1}(P)$ consists of an irreducible curve $\Gamma \subset \mathcal{W}$.
(3) $K_{\mathcal{W}} \cdot \Gamma < 0$.

Let $\mathrm{Exc}(F)$ be the exceptional loci of F. An extremal neighborhood is *flipping* if $\mathrm{Exc}(F) = \Gamma$. Otherwise, $\mathrm{Exc}(F)$ is two-dimensional, and F is called *divisorial*.

In the flipping case, $K_{\overline{\mathcal{W}}}$ is not $\mathbb{Q}$-Cartier. Then one attempts another type of birational modification. A *flip* of a flipping extremal neighborhood

$$F\colon (\Gamma \subset \mathcal{W}) \to (P \in \overline{\mathcal{W}})$$

is a proper birational morphism

$$F^+\colon (\Gamma^+ \subset \mathcal{W}^+) \to (P \in \overline{\mathcal{W}})$$

where $\mathcal{W}^+$ is normal with terminal singularities, $\mathrm{Exc}(F^+) = \Gamma^+$ is a curve, and $K_{\mathcal{W}^+}$ is $\mathbb{Q}$-Cartier and F^+-ample. A flip induces a birational map $\mathcal{W} \dashrightarrow \mathcal{W}^+$ to which we also refer as flip. When a flip exists then it is unique (cf. [KM1]). Mori [M4] proves that (3-fold) flips always exist.

A general $E \in |-K_{\mathcal{W}}|$ has only Du Val singularities [KM2]. Then a coarse classification goes like this: type A (semi-stable) and the remaining ones (exceptional). The exceptional ones were classified in [KM2], where everything is worked out. Semi-stable are more complicated, and they are divided into k1A and k2A. Mori completely shows the picture for k2A in [M3]. Later in [MP], Mori and Prokhorov classified the case k1A.

The semi-stable extremal neighborhoods can be seen as smoothings over $\mathbb{D}$ (see [HTU]), and so we have a Milnor fiber W_t for $\mathcal{W}$. Its second Betti number $b_2(W_t) \geq 1$ is an invariant. It turns out that for our purposes (running MMP for W-surfaces) it will be enough to consider the minimum Milnor number, i.e. $b_2(W_t) = 1$. The following is a definition/theorem (see [HTU, Prop. 2.1]).

Definition 4.6. Let $(P \in \overline{W})$ be a c.q.s. germ. Assume there is a partial resolution $f\colon W \to \overline{W}$ of $\overline{W}$ such that $f^{-1}(P)$ is a smooth rational curve Γ with one (two) Wahl singularity(ies) of W on it. Suppose $K_W \cdot \Gamma < 0$. Let $(W \subset \mathcal{W}) \to (0 \in \mathbb{D})$ be a $\mathbb{Q}$-Gorenstein smoothing of X over a smooth analytic germ of a curve $\mathbb{D}$. Let $(\overline{W} \subset \overline{\mathcal{W}}) \to (0 \in \mathbb{D})$ be the corresponding blowing down deformation of $\overline{W}$ [KM2, 11.4]. The induced birational morphism $(\Gamma \subset \mathcal{W}) \to (P \in \overline{\mathcal{W}})$ is called *extremal neighborhood of type mk1A (mk2A)*; we denote it by mk1A (mk2A).

Definition 4.7. A P-resolution $f^+\colon W^+ \to \overline{W}$ of a c.q.s. germ $(P \in \overline{W})$ is called *extremal P-resolution* if ${f^+}^{-1}(P)$ is a smooth rational curve Γ^+, and W^+ has only Wahl singularities (thus at most two; cf. [KSB, Lemma 3.14]).

Proposition 4.8. *Let $(\Gamma \subset \mathcal{W}) \to (P \in \overline{\mathcal{W}})$ be a flipping mk1A or mk2A, where $(\Gamma \subset W) \to (P \in \overline{W})$ is the contraction of Γ between the special fibers. Then there exists an extremal P-resolution $(\Gamma^+ \subset W^+) \to (P \in \overline{W})$, such that the flip $(\Gamma^+ \subset \mathcal{W}^+) \to (P \in \overline{\mathcal{W}})$ is obtained by the blowing down deformation of a $\mathbb{Q}$-Gorenstein smoothing of W^+. The commutative diagram of maps is*

$$\begin{array}{ccc} (\Gamma \subset \mathcal{W}) & \overset{flip}{\dashrightarrow} & (\Gamma^+ \subset \mathcal{W}^+) \\ & \searrow \quad \swarrow & \\ & (P \in \overline{\mathcal{W}}) & \\ & \downarrow & \\ & (0 \in \mathbb{D}), & \end{array}$$

and restricted to the special fibers we have

$$\begin{array}{ccc} (\Gamma \subset W) & \dashrightarrow & (\Gamma^+ \subset W^+) \\ & \searrow \quad \swarrow & \\ & (P \in \overline{W}). & \end{array}$$

Proof. [KM2, Sect.11 and Thm.13.5]. (See [M3, HTU] for explicit equations of the surfaces and 3-folds involved.) □

Proposition 4.9. *If an mk1A or mk2A is divisorial, then $(P \in \overline{W})$ is a Wahl singularity. The divisorial contraction $\mathcal{W} \to \overline{\mathcal{W}}$ induces the blowing down of a (-1)-curve between the smooth fibers of $\mathcal{W} \to \mathbb{D}$ and $\overline{\mathcal{W}} \to \mathbb{D}$.*

Proof. See [U3, Prop.2.8]. □

Next we show the numerical description of the W in an mk1A or in an mk2A (Definition 4.6), and of the W^+ in an extremal P-resolution (Definition 4.7). This description only requires toric computations on surfaces, the 3-folds $\mathcal{W}$ and $\mathcal{W}^+$ do not play a role. See more details in [HTU, §2].

($W \to \overline{W}$ for mk1A): Fix an mk1A with Wahl singularity $\frac{1}{n^2}(1, na-1)$. Let $\frac{n^2}{na-1} = [e_1, \ldots, e_s]$ be its continued fraction. Let $E_1, \ldots, E_s$ be the exceptional curves of the minimal resolution $\widetilde{W}$ of W with $E_j^2 = -e_j$ for all j. Notice that $K_W \cdot \Gamma < 0$ and $\Gamma \cdot \Gamma < 0$ imply that the strict transform of Γ in $\widetilde{W}$ is a (-1)-curve intersecting only one curve E_i transversally at one point. This data will be written as

$$[e_1, \ldots, \overline{e_i}, \ldots, e_s]$$

so that $\frac{\Delta}{\Omega} = [e_1, \ldots, e_i - 1, \ldots, e_s]$ where $0 < \Omega < \Delta$, and $(P \in \overline{W})$ is $\frac{1}{\Delta}(1, \Omega)$. Let $\beta_i, \alpha_i, \gamma_i$ be the numbers Definition 2.12 for the singularity $\frac{1}{n^2}(1, na-1)$. Then

$$\Delta = n^2 - \beta_i \alpha_i \qquad \Omega = na - 1 - \gamma_i \beta_i$$

and, if $\delta := \frac{\beta_i + \alpha_i}{n}$, we have $K_W \cdot \Gamma = -\frac{\delta}{n} < 0$ and $\Gamma \cdot \Gamma = -\frac{\Delta}{n^2} < 0$.

($W \to \overline{W}$ for mk2A): Consider now an mk2A with Wahl singularities
$$\frac{1}{n_0^2}(1, n_0a_0 - 1), \frac{1}{n_1^2}(1, n_1a_1 - 1).$$
Let $E_1, \ldots, E_{s_0}$ and $F_1, \ldots, F_{s_1}$ be the exceptional divisors over $\frac{1}{n_0^2}(1, n_0a_0 - 1)$ and $\frac{1}{n_1^2}(1, n_1a_1 - 1)$ respectively, such that $\frac{n_0^2}{n_0a_0-1} = [e_1, \ldots, e_{s_0}]$ and $\frac{n_1^2}{n_1a_1-1} = [f_1, \ldots, f_{s_1}]$ with $E_i^2 = -e_i$ and $F_j^2 = -f_j$. We know that the strict transform of Γ in the minimal resolution $\widetilde{W}$ of W is a (-1)-curve intersecting only E_{s_0} and F_1 transversally at one point each. The data for mk2A will be written as
$$[e_1, \ldots, e_{s_0}] - [f_1, \ldots, f_{s_1}],$$
and
$$\frac{\Delta}{\Omega} = [e_1, \ldots, e_{s_0}, 1, f_1, \ldots, f_{s_1}]$$
where $0 < \Omega < \Delta$ and $(P \in \overline{W})$ is $\frac{1}{\Delta}(1, \Omega)$.

We define $\delta := n_0a_1 - n_1a_0$ [9], and so
$$\Delta = n_0^2 + n_1^2 - \delta n_0n_1, \quad \Omega = (n_0 - \delta n_1)a_0 + n_1a_1 - 1.$$
We have $K_W \cdot \Gamma = -\frac{\delta}{n_0n_1} < 0$ and $\Gamma \cdot \Gamma = -\frac{\Delta}{n_0^2n_1^2} < 0$.

($W^+ \to \overline{W}$): In analogy to an mk2A, an extremal P-resolution has data
$$[e_1, \ldots, e_{s_0}] - c - [f_1, \ldots, f_{s_1}],$$
so that
$$\frac{\Delta}{\Omega} = [e_1, \ldots, e_{s_0}, c, f_1, \ldots, f_{s_1}]$$
where $-c$ is the self-intersection of the strict transform of Γ^+ in the minimal resolution of W^+, $0 < \Omega < \Delta$, and $(P \in \overline{W})$ is $\frac{1}{\Delta}(1, \Omega)$. As for an mk2A , here $\frac{n'^2_0}{n'_0a'_0-1} = [e_1, \ldots, e_{s_0}]$ and $\frac{n'^2_1}{n'_1a'_1-1} = [f_1, \ldots, f_{s_1}]$. If a Wahl singularity (or both) is (are) actually smooth, then we set $n'_1 = a'_1 = 1$ and/or $n'_0 = 1$ and $a'_0 = 0$. We define
$$\delta = (c-1)n'_0n'_1 + n'_1a'_0 - n'_0a'_1,$$
and so $\Delta = n'^2_0 + n'^2_1 + \delta n'_0n'_1$ and, when both $n'_i \neq 1$,
$$\Omega = -n'^2_1(c-1) + (n'_0 + \delta n'_1)a'_0 + n'_1a'_1 - 1.$$
(One easily computes Ω when one or both $n'_i = 1$.) We have
$$K_{W^+} \cdot \Gamma^+ = \frac{\delta}{n'_0n'_1} > 0 \quad \text{and} \quad \Gamma^+ \cdot \Gamma^+ = -\frac{\Delta}{n'^2_0n'^2_1} < 0.$$
How to compute explicitly? First we recall Mori's algorithm to compute the numerical data of either the flip or the divisorial contraction for any mk2A; cf. [M3].

Let us consider an arbitrary extremal neighborhood $\mathbb{E}$ of type $mk2A$ with numerical data (m, b), (n, a), so that the Wahl singularities are
$$\frac{1}{m^2}(1, mb - 1), \;\; \frac{1}{n^2}(1, na - 1),$$

[9] If you want to use it in the presence of a smooth point, then we set either $n_1 = a_1 = 1$, or $n_0 = 1$ and $a_0 = 0$.

$\delta = ma + nb - mn > 0$, and $0 < \Omega < \Delta$ as above. Without loss of generality, we assume $n > m$. (Using the formulas for δ and Δ, it is easy to see that $m \neq n$.) From this data, Mori constructs other extremal neighborhoods $\mathbb{E}'$ of type mk2A such that both $\mathbb{E}$ and $\mathbb{E}'$ are of the same type (either flipping or divisorial), and after the birational modification the corresponding central fibers are the same. We now explain how to find these $\mathbb{E}'$, and Mori's criterion to know when $\mathbb{E}$ is flipping or divisorial.

Assume $\delta > 1$, the case $\delta = 1$ will be treated separately.

Let us define the recursion $\zeta_1 = 0$, $\zeta_2 = 1$, $\zeta_{i+1} + \zeta_{i-1} = \delta\zeta_i$, for $i \geq 2$. We note that the recursion defines the infinite continued fraction

$$\frac{\delta + \sqrt{\delta^2 - 4}}{2} = \delta - \cfrac{1}{\delta - \cfrac{1}{\ddots}}.$$

One can show that

$$\left(\zeta_{i+1}n - \zeta_i m, \zeta_{i+1}a - \zeta_i b\right) \tag{4.1}$$

is a pair of positive integers for all $i \geq 1$. But one can prove that there exists an integer $i_0 \geq 1$ such that

$$\left(\zeta_{i+1}m - \zeta_i n, \zeta_{i+1}b - \zeta_i a\right) \tag{4.2}$$

is a pair of positive integers only for $1 \leq i \leq i_0 - 1$. Precisely, we have $\zeta_{i_0+1}m - \zeta_{i_0}n \leq 0$. Two consecutive pairs of positive numbers of the form (4.1) or (4.2) above define the two Wahl singularities of an $\mathbb{E}'$, with associated numbers δ, Ω, and Δ (same numbers as for $\mathbb{E}$). Below we will show precisely the $\mathbb{E}'$. Mori [M3] proves:

Theorem 4.10. *The extremal neighborhood $\mathbb{E}$ is of flipping type if and only if $\zeta_{i_0+1}m - \zeta_{i_0}n < 0$. Otherwise (i.e. $\zeta_{i_0+1}m - \zeta_{i_0}n = 0$) $\mathbb{E}$ is of divisorial type.*

We note that this procedure gives an initial $\mathbb{E}'$, right before reaching the index i_0. We call it the *initial* mk2A associated to a given $\mathbb{E}$.

From an initial mk2A $\mathbb{E}_1 := \mathbb{E}'$ with Wahl singularities defined by pairs (n_0, a_0) and (n_1, a_1) with $n_0 < n_1$, and numbers δ, Δ and Ω, where $\delta n_0 - n_1 \leq 0$. We also allow the mk1A special case $n_0 = a_0 = 1$.

For $i \geq 1$, we have the Mori recursions (see [HTU, §3.3])

$$n(0) = n_0, \quad n(1) = n_1, \quad n(i-1) + n(i+1) = \delta n(i)$$

and $a(0) = a_0$, $a(1) = a_1$, $a(i-1) + a(i+1) = \delta a(i)$. When $\delta > 1$, for each $i \geq 1$ we have an mk2A $\mathbb{E}_i$ with Wahl singularities defined by the pairs $(n(i), a(i)), (n(i+1), a(i+1))$. We have $n(i+1) > n(i)$. The numbers δ, Δ and Ω, and the flipping or divisorial type of $\mathbb{E}_i$ are equal to the ones associated to $\mathbb{E}_1$. We call this sequence of mk2As a *Mori sequence*. If $\delta = 1$, then the initial mk2A must be flipping (by Mori's criterion), and the Mori sequence above gives only one more mk2A with data $n_2 = n_1 - n_0, a_2 = a_1 - a_0$ and n_1, a_1.

As a summary, from the numerical data of $\mathbb{E}_1$ we have according to $\delta n_0 - n_1$:

(=0) (see [HTU, Prop.3.13]) Divisorial type: then $n_0 = \delta$, $n_1 = \delta^2 = \Delta$, $a_1 = \delta a_0 - 1 = \Omega$.

(<0) (see [HTU, Prop.3.15, Thm.3.20]) Flipping type: the extremal P-resolution W^+ has $n'_0 = n_1 - \delta n_0$, $a'_0 = a_1 - \delta a_0 - (c-1)n'_0$, and $n'_1 = n_0,\ a'_1 = a_0$, where $c - 1 \geq 0$ is the adequate multiple, and c is the self-intersection of the proper transform of Γ^+ in the minimal resolution.

In [HTU] we have the following method to compute for all extremal neighborhoods of type mk1A. It is proved that a given exceptional neighborhood of type mk1A degenerates to two mk2A sharing the type, and the central fiber of the resulting birational operation.

Proposition 4.11. [HTU, §2.3 and §3.4] *Let $[e_1, \ldots, \overline{e_i}, \ldots, e_s]$ be the data of an mk1A with $\frac{n^2}{na-1} = [e_1, \ldots, e_s]$. Let δ, Δ, Ω be as in the above numerical description of an mk1A.*

Let $\frac{n_2}{a_2} = [e_1, \ldots, e_{i-1}]$ and $\frac{n_1}{n_1-a_1} = [e_s, \ldots, e_{i+1}]$, if possible (this is, for the first $i > 1$, for the second $i < s$). Then, there are mk2A with data

$$[f_1, \ldots, f_{s_2}] - [e_1, \ldots, e_s] \quad \textit{and} \quad [e_1, \ldots, e_s] - [g_1, \ldots, g_{s_1}],$$

where $\frac{n_2^2}{n_2a_2-1} = [f_1, \ldots, f_{s_2}]$, $\frac{n_1^2}{n_1a_1-1} = [g_1, \ldots, g_{s_1}]$, such that the corresponding cyclic quotient singularity $\frac{1}{\Delta}(1, \Omega)$ and δ are the same for the mk1A and the mk2A. Moreover, each of the mk2A deforms (over a smooth analytic germ of a curve) to the mk1A by $\mathbb{Q}$-Gorenstein smoothing up $\frac{1}{n_i^2}(1, n_ia_i - 1)$ while keeping $\frac{1}{n^2}(1, na-1)$, and there are two possibilities: either these three extremal neighborhoods are

(1) flipping, with the same extremal P-resolution for the flip, or
(2) divisorial, with the same $(P \in \overline{W})$.

In [HTU] we show that this a key to give a complete description which provides a universal family for both flipping and divisorial contractions; see [HTU, §3]. A flip which appears frequently in calculations is the following

Proposition 4.12. *Let $[e_1, \ldots, e_{s-1}, \overline{e_s}]$ be an mk1A. Then it is of flipping type. Let $i \in \{1, \ldots, s\}$ be such that $e_i \geq 3$ and $e_j = 2$ for all $j > i$. (If $e_s > 2$, then we set $i = s$.)*

Then the data for W^+ is $e_1 - [e_2, \ldots, e_i - 1]$.

Example 4.13. (Divisorial family) Consider the Wahl singularity $(P \in \overline{W}) = \frac{1}{4}(1, 1)$. So $\Delta = 4$ and $\Omega = 1$, and $\delta = 2$. Then the numerical data of any mk1A and any mk2A of divisorial type associated to $(P \in \overline{W})$ can be read from the Mori train

$$[4] - [2, \bar{2}, 6] - [2, 2, 2, \bar{2}, 8] - [2, 2, 2, 2, 2, \bar{2}, 10] - \cdots$$

Notice that $\delta = 2$. For example, $[2, 2, 2, 2, 2, \bar{2}, 10]$ is an mk1A , and $[2, \bar{2}, 6] - [2, 2, 2, \bar{2}, 8]$ is an mk2A.

Example 4.14. (Flipping family) Let $\frac{1}{11}(1, 3)$ be the cyclic quotient singularity $(P \in \overline{W})$. So $\Delta = 11$ and $\Omega = 3$. Consider the extremal P-resolution $W^+ \to \overline{W}$ defined by $[4] - 3$. Here $n'_1 = a'_1 = 1$, $n'_0 = 2$, $a'_0 = 1$, $\delta = 3$, and the "middle" curve is a (-3)-curve. Then the numerical data of any mk1A and any mk2A associated to W^+ can be read from the Mori trains

$$[\bar{2}, 5, 3] - [2, 3, \bar{2}, 2, 7, 3] - [2, 3, 2, 2, 2, \bar{2}, 5, 7, 3] - \cdots$$

and

$$[4] - [2, \bar{2}, 5, 4] - [2, 2, 3, \bar{2}, 2, 7, 4] - [2, 2, 3, 2, 2, 2, \bar{2}, 5, 7, 4] - \cdots$$

These two Mori sequences provide the numerical data of the universal antiflip of $[4] - 3$. For particular examples, we have that $[2, 3, \bar{2}, 2, 7, 3]$ and $[2, \bar{2}, 5, 4]$ are mk1A whose flips have W^+ as central fiber.

In this link there is a computer algorithm to easily compute flips, divisorial contractions, and the families of them RUN MMP [V4].

Exercises.

(1) Prove Proposition 4.12 (The usual flip).

(2) Let $0 < a < \delta$ be coprime numbers. Show that the Mori train for the divisorial contractions over $\frac{1}{\delta^2}(1, \delta a - 1)$ starts with

$$\left[\begin{pmatrix} \delta^2 \\ \delta a - 1 \end{pmatrix}\right] - (1) - \left[\begin{pmatrix} \delta \\ a \end{pmatrix}\right] \to \frac{1}{\delta^2}(1, \delta a - 1).$$

(3) Sometimes a flipping contraction $(\Gamma^- \subset W^-) \to (P \in \overline{W})$ is over a Wahl singularity $(P \in \overline{W})$. Give examples for each Wahl singularity.

(4) Let $W^- \dashrightarrow W^+$ be a flip at the level of surfaces. Verify that the indices of the singularities either stay the same or decrease (including singularities that become smooth points), and that at least one of them strictly decreases. This is significant to know that the MMP stops.

(5) Some Mori trains appear in nature. In [UZ1], it is proved that the color Markov branches (see Figure 2) can be explained as Mori trains for suitable c.q.s. which emerge deeper and deeper in the MMP as we change colors. For example, the Markov train corresponding to $\frac{1}{5}(1, 1)$ is precisely the Fibonacci branch. Check this (see [UZ1, Section 7.1]).

(6) $\star$ Mori trains are formed by wagons which contain Wahl chains. To go from one wagon to the next, we can contract the (-1)-curve representing the bar, and then blow-up several times over one of the two nodes in the corresponding exceptional curve, until one reaches the new Wahl chain. Find out a combinatorial algorithm at the level of Wahl chains that constructs the Mori train. The following is an example of the first 8 wagons of a Mori train for the P-resolution $[2, 5] - 1 - [3, 2, 6, 2]$.

$[3, 2, 6, 2]$
$[3, 2, 2, \bar{2}, 7, 2, 2, 6, 2]$
$[3, 2, 2, 2, 5, \bar{2}, 2, 2, 2, 9, 2, 2, 6, 2]$
$[3, 2, 2, 2, 5, 2, 2, 2, 2, 2, \bar{2}, 7, 2, 2, 9, 2, 2, 6, 2]$
$[3, 2, 2, 2, 5, 2, 2, 2, 2, 2, 2, 5, \bar{2}, 2, 2, 2, 9, 2, 2, 9, 2, 2, 6, 2]$
$[3, 2, 2, 2, 5, 2, 2, 2, 2, 2, 2, 5, 2, 2, 2, 2, 2, \bar{2}, 7, 2, 2, 9, 2, 2, 9, 2, 2, 6, 2]$
$[3, 2, 2, 2, 5, 2, 2, 2, 2, 2, 2, 5, 2, 2, 2, 2, 2, 2, 5, \bar{2}, 2, 2, 2, 9, 2, 2, 9, 2, 2, 9, 2, 2, 6, 2]$
$[3, 2, 2, 2, 5, 2, 2, 2, 2, 2, 2, 5, 2, 2, 2, 2, 2, 2, 5, 2, 2, 2, 2, 2, \bar{2}, 7, 2, 2, 9, 2, 2, 9, 2, 2, 9, 2, 2, 6, 2]$

(7) $\star\star$ Given an $mk1A$ or $mk2A$ of flipping type, then the flip at the level of the surface W^+ is the $\mathbb{Q}$-Gorenstein smoothing of an extremal P-resolution of some c.q.s. This smoothing is very particular, and it is described in [HTU]. Not every smoothing comes from a flip (or it has an anti-flip). In [HTU, Question 1.3] it is asked to classify all $\mathbb{Q}$-Gorenstein smoothings which admit a not necessarily terminal anti-flip. In the recent thesis work of Arié Stern [S6], it is shown an answer for particular situations. It is open in general. See [HTU, Example 3.26] for a canonical non-terminal anti-flip.

4.3. **MMP for W-surfaces II.**

The base for this section is [U4, Section 2]. The goal is to run MMP on a W-surface, and to show what the ending results share an analogy with the classical (Italian) case of a nonsingular projective surface. We recall that for nonsingular surfaces, after contracting finitely many (-1)-curves, we arrive to either a unique minimal model (here canonical class is nef), or a geometrically ruled surface $\mathbb{P}_C(\mathcal{F})$ where C is a nonsingular curve and $\mathcal{F}$s a rank two vector bundle on C, or $\mathbb{P}^2$. *What do we get for W-surfaces?*

Let W be a W-surface. We recall that W is minimal if K_W is nef (and then all smooth fibers have K_{W_t} nef; If K_W is ample, then K_{W_t} is ample for all t as well). Assume that K_W is not nef. Then (by for example [KK2, 2.1.1]) there is a K-extremal ray $\mathbb{P}^1 =: \Gamma \subset W$ such that $\Gamma \cdot K_W < 0$. We have the following options:

(I) If $\Gamma^2 > 0$, then $\text{Pic}(W)$ has rank 1 and $-K_W$ is ample [KK2, 2.3.3]. Hence $-K_{W_t}$ is ample for any t [KM1, Prop.1.41], and so W_t is rational for any t. Moreover, the rank 1 condition implies that $\chi_{\text{top}}(W_t) = 3$ for all t, and so W_t is isomorphic to $\mathbb{P}^2$. This type of degenerations of $\mathbb{P}^2$ were classified in [B5, M1, HP]. According to [HP, Cor.1.2], the surface W must be a $\mathbb{Q}$-Gorenstein deformation of a weighted projective plane $\mathbb{P}(a^2, b^2, c^2)$ where (a, b, c) satisfies the Markov equation $a^2 + b^2 + c^2 = 3abc$. We will say more at the end, and this is the connection with Section 1.2.

(II) If $\Gamma^2 = 0$, then there is a fibration $h\colon W \to B$ with irreducible fibers and general fiber isomorphic to $\mathbb{P}^1$ [KK2, 2.3.3]. Let $\tilde{h}\colon \tilde{W} \to B$ be the corresponding fibration on the minimal resolution $\tilde{W}$ of W. Then, over a $b \in B$ where the fiber has a Wahl singularity, the fiber in $\tilde{W}$ has two possible configuration types; see [HP, Prop.7.4]. It is a simple check that none of them is possible when the singularities are of Wahl type. Therefore, $(W \subset \mathcal{W}) \to (0 \in \mathbb{D})$ is a smooth deformation of a geometrically ruled surface W.

(III) If $\Gamma^2 < 0$, then we can apply to $(W \subset \mathcal{W}) \to (0 \in \mathbb{D})$ a birational transformation defined by an extremal neighborhood of type mk1A or mk2A of flipping or divisorial type [HTU, Thm.5.3]. After that we arrive to a new W-surface $(W^+ \subset \mathcal{W}^+) \to (0 \in \mathbb{D})$.

Remark 4.15. It turns out that minimal models are unique meaning the surface W and all fibers are unique up to isomorphism. But the families may not be isomorphic over $\mathbb{D}$. In [U4, Prop.2.6] it is proved the uniqueness, but it is wrongly stated that families (the 3-folds) are isomorphic. Indeed for this type of families we may have flops, and they are produced by the situation of an M-resolution of a T-singularity. To have an example in mind check what happens in the Atiyah flop when we see it as a family over $\mathbb{C}$. In this family one sees two ways to modify so that fibers are isomorphic but 3-folds are not. We call them *Kawamata flops* as they were used in [K4] to show a semi-orthogonal decomposition of the derived category of the smoothing.

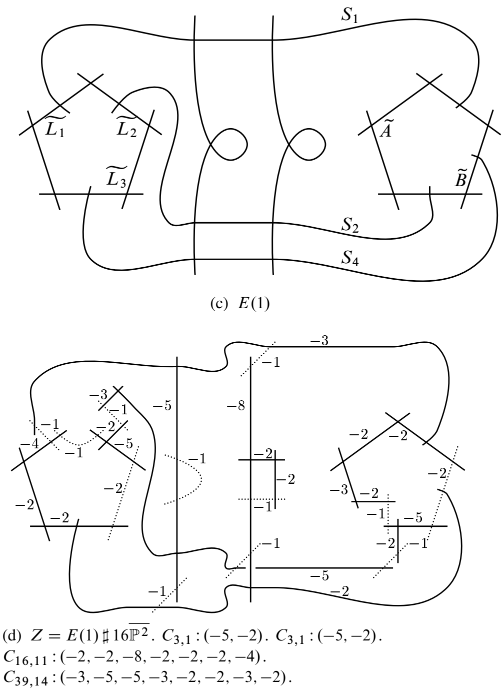
Example 4.16. Let us run MMP in the example in Figure 15. In this example, we have a W with four Wahl singularities, whose Wahl chains are shown in the minimal resolution $\tilde{W}$ of W. In that picture there is a curve that is negative for the canonical class. It gives the data of a flipping mk2A, and it is the image of the (-1)-curve between the two Wahl chains shown in Figure 16, which also shows the corresponding flip.

We then flip this curve in W to obtain a surface W^+ whose minimal resolution is shown in Figure 17. One can prove that W^+ has nef canonical class [U3, Section 7].

Let us come back to the general setup.

Let W be a minimal W-surface with W indeed singular. *What can we say about the possible W_t?* Let us fix the Kodaira dimension of W_t, $t \neq 0$. Then:

Kodaira 0: Only Enriques surfaces W_t can appear with W singular [K2, Theorem 4.1]. These degenerations of Enriques surfaces are studied in [U2].



(c) $E(1)$

(d) $Z = E(1)\sharp 16\overline{\mathbb{P}^2}$. $C_{3,1}: (-5,-2)$. $C_{3,1}: (-5,-2)$. $C_{16,11}: (-2,-2,-8,-2,-2,-2,-4)$. $C_{39,14}: (-3,-5,-5,-3,-2,-2,-3,-2)$.

FIGURE 15. The wrong example from [PPS1]

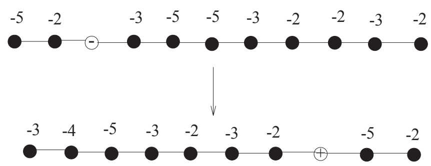


FIGURE 16. The flip that fixes the example

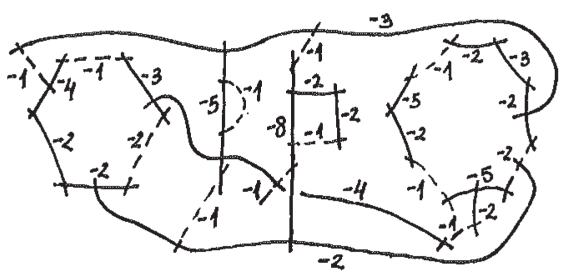


FIGURE 17. Minimal resolution after the flip

Kodaira 1: Kawamata classifies all possible degenerations in [K2, Theorem 4.2]. Both the W_t and W have an elliptic fibration compatible with the deformation. See also [U3, Section 4] for the case when W_t is a Dolgachev surface.

Kodaira 2: Here there are just too many possibilities for W_t which has K_{W_t} big and nef. It is not understood how to predict the existence of these degenerations. The canonical model of W has only ADE and T-singularities, and the corresponding family over $\mathbb{D}$ is a curve germ in the KSBA moduli space of surfaces of general type.

For K_W big and nef, *Is it possible to classify the possible W?*

Let S be a minimal model of a minimal resolution of W. Then we have the following options for S [RU1, Prop.2.3]:

1. It is rational.
2. It is a K3 or an Enriques surface.
3. It has Kodaira dimension 1 and $q(S) = 0$.
4. It is of general type with $q(S) = 0$ (Proposition 4.17) and $K_S^2 < K_W^2$.

In particular, we have singular minimal models only for regular varieties (i.e. $q = 0$)!

This shows a hierarchy between W_t of general type and W, which is either based on K_S^2 or the Kodaira dimension of S. There are just too many and unclassifiable cases for each alternative. Even if we fix extra invariants such as K^2.

For example, if $K^2 = 1$, then there are no surfaces of general type in the boundary. Also singularities of W are significantly bounded, when S is not rational [RU1].

Proposition 4.17. *If the surface S above is of general type, then $q(S) = 0$. In this way W-surfaces of general type have no irregularity.*

Proof. As always, let $\pi\colon X \to W$ be the minimal resolution of W, and let $\sigma\colon X \to S$ the composition of blow ups from S. We have $\pi^*(K_W) \equiv \sigma^*(K_S) + \sum_j a_i E_i + \sum_j b_j C_j$, where E_i are the exceptional curves for σ and C_j are the exceptional curves for π. Both a_i and b_j are positive.

In this proof we will use the Albanese variety Alb(S) of S; Cf. [B1, V]. Assume $q(S) > 0$. Let $\alpha\colon S \to$ Alb(S) be the Albanese map. Its image can be a curve or a surface.

Let us assume that its image is a curve. Then $\alpha\colon S \to \alpha(S)$ is a fibration into a nonsingular curve of genus $q(S)$ [B1, V]. Then we have a fibration $X \to \alpha(S)$, and so all the C_i are inside of fibers. In particular, we can rewrite $\pi^*(K_W) \equiv \sigma^*(K_S) + \sum_j a_i E_i + \sum_j b_j C_j$ as

$$\pi^*(K_W) \equiv \sigma^*(K_S) + \sum_i \sum_j c_{i,j} G_{i,j}$$

where $\sum_j G_{i,j}$ is inside of a fiber and $c_{i,j} > 0$. Then we square to find $0 < K_W^2 = K_S^2 - \sum_i N_i$, where $N_i \geq 0$ by Zariski's lemma [BHPVdV, III Lemma 8.2]. But $K_W^2 > K_S^2$.

Let us assume now that the image is a surface. Then we use Stein factorization for $\alpha\colon S \to \alpha(S)$ via $\beta\colon S \to Y$ birational map, and $Y \to \alpha(S)$ finite map. Now consider the composition $X \to Y$, which is a birational map and the images of the C_i must be contracted as $\alpha(S)$ cannot have any rational curve. But then analog argument of the previous paragraph works in this situation. □

Let us elaborate better the ending case when $\Gamma^2 > 0$. We know that W_t must be $\mathbb{P}^2$. So, one may wonder in full generality: *What are all the normal degenerations of $\mathbb{P}^2$?*

Bădescu [B5] initiated this quest for rational surfaces. Following ideas in [B5], Manetti shows in [M1, Main Theorem] that if the normal degeneration has only quotient singularities, then it is a W-surface. (*What about normal degenerations to other rational singularities, such as generalized Wahl?* Do it! See [M1, Theorem 11]). Hacking and Prokhorov not only classify these W-surfaces, but also fully work out the case when W has Picard number 1 and $-K_W$ is ample (with T-singularities; if we allow only Wahl singularities, then this can only be $\mathbb{P}^2$; see for example [HP, Table 1]). (Later Prokhorov [P6] did it for log canonical singularities.)

The picture for $\mathbb{P}^2$ is shown in Figure 18, where in the way between two weighted projective planes we have Ws with one or two singularities (from the original weighted projective planes). To go from one to the other one uses the mutations for solutions to the Markov equation.

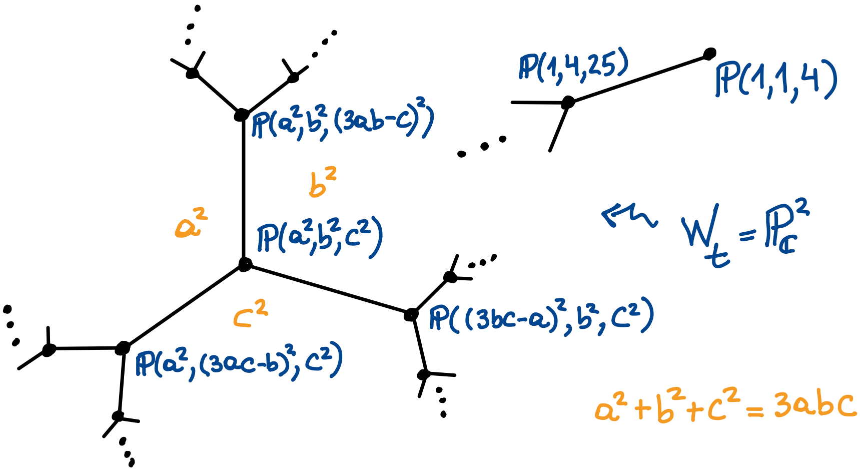


FIGURE 18. c.q.s. degenerations of $\mathbb{P}^2$.

In [U4, Section 3] it is shown that we can always obtain a W-surface with birational fibers from a smooth W-surface, by applying finitely many anti-flips and/or divisorial contractions (up and down). In particular, in [U4] is proved that any degeneration of $\mathbb{P}^2$ can be understood from a deformation of $\mathbb{F}_1$ into $\mathbb{F}_3$ (one singularity), $\mathbb{F}_5$ (two singularities), or $\mathbb{F}_7$ (three singularities. See [U4, Figure 1] (Figure 19) for an example.

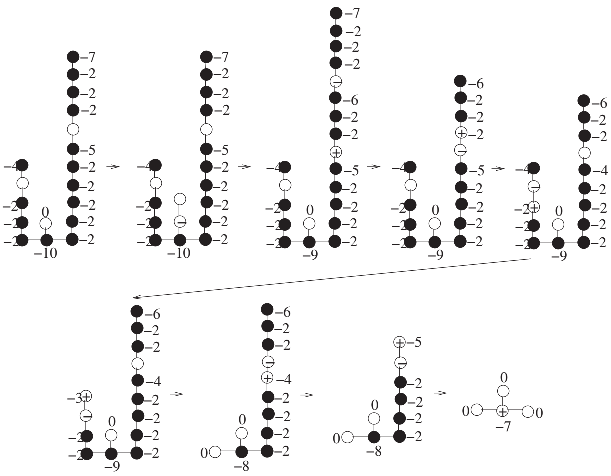


FIGURE 19. How to see the degeneration of $\mathbb{P}^2$ into $\mathbb{P}(2^2, 5^2, 29^2)$ as coming from a deformation of $\mathbb{F}_1$ into $\mathbb{F}_7$.

Exercises.

(1) Compute $K \cdot \Gamma$ and Γ^2 for the bad curve C in Figure 15, which is used in the flip Figure 16. Show that indeed K_{W^+} is big and nef.

(2) Find W-surfaces where W_t is an Enriques surface, and W_0 has Wahl singularities different than $\frac{1}{4}(1,1)$.

(3) Let W be a W-surface with general fiber $\mathbb{F}_n$. Show that after running MMP we have either a Markov degeneration of $\mathbb{P}^2$ or a deformation with central fiber $\mathbb{F}_m$ with $m > n$.

(4) $\star$ By running MMP, classify W-surfaces when W is birational to W_t for some $t \neq 0$.

(5) $\star$ Show that a minimal W-surface with W_t K3 surfaces must be smooth. Show that without the minimality condition we can have singular W-surfaces.

(6) $\star$ Given a degeneration of $\mathbb{P}^2$ into a surface W as in Hacking-Prokhorov, we can blow-up a general point for all fibers and obtain a similar degeneration of $\mathcal{F}_1$ into $\text{Bl}_p(W)$. Then we can run MMP starting with a unique flipping curve. This MMP is unique, in the sense that at each step we have always a unique flip (see the example in Figure 19). This is explained in [UZ1]. Now consider arbitrary $\mathbb{Q}$-Gorenstein degenerations of Hirzebruch surfaces with only Wahl singularities. Are there any Markov sort of moduli spaces? Is it possible to describe the corresponding MMP?

(7) $\star\star$ Are there normal degenerations of $\mathbb{P}^2$ with non quotient singularities? Find a classification.

Summary of MMP for W-surfaces and classic nonsingular projective surfaces.

<table>
<tr><th>Modern
$W_t \rightsquigarrow W$ W-surface</th><th>Classic
X = nonsingular surface</th></tr>
<tr><td>intersection theory $|_{\mathbb{D}}$
K^2, χ_{top}, p_g, q constant</td><td>intersection theory
K^2, χ_{top}, p_g, q</td></tr>
<tr><td>MMP relative to $\mathbb{D}$

K_W nef $\Rightarrow$ K_{W_t} nef and W is unique

K_W no nef $\Rightarrow \exists\, \Gamma = \mathbb{P}^1$ with $K_W \cdot \Gamma < 0$.
If $\Gamma^2 < 0 \Rightarrow$
Flip or Div. Contr.
<table>
<tr><td></td><td>Flip</td><td>Div. Contr.</td></tr>
<tr><td>W</td><td>changes to $\Gamma^+ \subset W^+$ birationally</td><td>$W \to W'$ blow-down to Wahl sing.</td></tr>
<tr><td>W_t</td><td>does not change</td><td>$W_t \to W'_t$ blow-down of (-1)-curve</td></tr>
</table>
If no such curves $\Rightarrow$
$\Gamma^2 = 0$ is $\mathbb{P}_C(\mathcal{E}) \rightsquigarrow \mathbb{P}_C(\mathcal{E}')$ as
or $\Gamma^2 > 0 \Rightarrow W_t \rightsquigarrow \mathbb{P}^2$ (MARKOV)</td>
<td>MMP = Castelnuovo

K_X nef $\Rightarrow$ unique

K_X no nef $\Rightarrow \exists\, \Gamma = \mathbb{P}^1$ with $K_X \cdot \Gamma < 0$.
If $\Gamma^2 < 0 \Rightarrow$ (-1)-curve & Blow-down
$X \longrightarrow X'$
$\Gamma \mapsto p$ = nonsing.

If no such curve, then
$\Gamma^2 = 0 \Rightarrow \mathbb{P}_C(\mathcal{E}) = X$
C = curve, $\mathcal{E}$ rank 2 v.b.
or $\Gamma^2 > 0 \Rightarrow X = \mathbb{P}^2$.</td></tr>
<tr><td>K_W big nef $\Rightarrow \exists\, W$ can with T-singularities</td><td>K_X big nef $\Rightarrow \exists\, X$ can with Du Val sing.</td></tr>
</table>

5. N-RESOLUTIONS

N-resolutions are the negative analogues to the M-resolutions of a cyclic quotient singularity. They were defined in [TU] to construct interesting exceptional collections of vector bundles in W-surfaces. We first review the basics of N-resolutions, and in the next section, we work out exceptional collections. There are no any other applications to N-resolution yet.

5.1. **Existence and uniqueness.**

Lat $0 < \Omega < \Delta$ be coprime integers. We have the c.q.s. $(P \in \overline{W}) = \frac{1}{\Delta}(1, \Omega)$, and the HJ continued fraction

$$\frac{\Delta}{\Omega} = e_1 - \cfrac{1}{e_2 - \cfrac{1}{\ddots - \frac{1}{e_\ell}}},$$

and its dual $\frac{\Delta}{\Delta - \Omega} = [b_1, \ldots, b_s]$. We saw that the set of P-resolutions (Definition 3.6) is in bijection with the zero continued fractions in

$$K(\overline{W}) = \{[k_1, \ldots, k_s] = 0 \colon \text{ such that } 1 \le k_i \le b_i\},$$

and it is also in bijection with the set of M-resolutions (Definition 3.11). Each M-resolution of $(P \in \overline{W})$ looks like Figure 20, and they are part of the following more general definition.

Definition 5.1. A *Wahl-resolution* $(\Gamma_1 \cup \ldots \cup \Gamma_r \subset W)$ is a surface germ that contains a chain of smooth projective rational curves $\Gamma_1, \ldots, \Gamma_r$ that are toric boundary divisors at Wahl singularities $P_0, \ldots, P_r$ (as in Figure 20), the surface is smooth elsewhere (we also allow P_i to be smooth points). We choose a toric boundary divisor germ Γ_0 at P_0 complementary to Γ_1 and Γ_{r+1} at P_r complementary to Γ_r. In addition, we assume that it admits a contraction $(\Gamma_1 \cup \ldots \cup \Gamma_r \subset W) \to (P \in \overline{W})$.

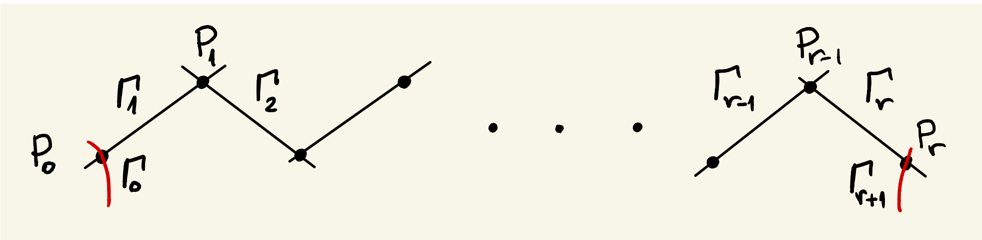


FIGURE 20. A Wahl resolution with r curves Γ_i

Notation 5.2. The singularity P_i of a Wahl-resolution W has type $\frac{1}{n_i^2}(1, n_i a_i - 1)$, where the Hirzebruch-Jung continued fraction of $\frac{n_i^2}{n_i a_i - 1}$ goes in the direction from Γ_i to Γ_{i+1}. For smooth points, $n_i = a_i = 1$. We note that

$$\Gamma_i^2 + K_W \cdot \Gamma_i = -\frac{1}{n_i^2} - \frac{1}{n_{i-1}^2}$$

for $i = 1, \ldots, r$. Let $\delta_i := n_{i-1} n_i |K_W \cdot \Gamma_i|$ (a non-negative integer).

The purpose of this section is to define the N-resolutions of $(P \in \overline{W})$ [TU]. The set of N-resolutions will be in bijection with the set of M-resolutions, and a particular characteristic of an N-resolution is that intersections of exceptional curves with the canonical class are nonpositive.

Definition 5.3. Let W^+ be an M-resolution of a c.q.s. $(P \in \overline{W})$. The corresponding *N-resolution* W^- is a Wahl-resolution of $(P \in \overline{W})$ with curves $\bar{\Gamma}_i$ and singularities $\bar{P}_i$ of type $\frac{1}{\bar{n}_i^2}(1, \bar{n}_i\bar{a}_i - 1)$ for $i = 0, \ldots, r$ such that

(1) The singularity $\bar{P}_r$ is P_0 with $\bar{n}_r = n_0$ and $\bar{a}_r = a_0$. Furthermore, for every $i = 1, \ldots, r$, the contraction of $\bar{\Gamma}_{r-i+1} \cup \ldots \cup \bar{\Gamma}_r \subset W^-$ is a c.q.s isomorphic to the contraction of $\Gamma_1 \cup \ldots \cup \Gamma_i \subset W^+$. We denote that c.q.s. by $\frac{1}{\Delta_i}(1, \Omega_i)$.
(2) $\bar{\delta}_{r-i+1} = \delta_i$ for $i = 1, \ldots, r$.
(3) $-K_{W^-}$ is relatively nef, i.e., $K_{W^-} \cdot \bar{\Gamma}_i \le 0$ for $i = 1, \ldots, r$.

Notation 5.4. Suppose the M-resolution $W^+ \to \overline{W}$ corresponds to a zero-fraction

$$[k_1, \ldots, k_s] \in K(\overline{W}).$$

The surface W^+ contains curves $\Gamma_1, \ldots, \Gamma_r$ and Wahl singularities at P_i of type $\frac{1}{n_i^2}(1, n_ia_i - 1)$. We have $\delta_i = n_{i-1}n_iK_{W^+} \cdot \Gamma_i \ge 0$ for all $i = 1, \ldots, r$. Let $d_i := b_i - k_i \ge 0$. We have $d_1 + \ldots + d_s = r + 1$. Let $d_{i_1}, \ldots, d_{i_e}$ be the set of nonzero d_i with $i_1 < i_2 < \ldots < i_e$.

Proposition 5.5. *The numbers $\delta_1, \ldots, \delta_r$ can be computed as follows: for $k = 1, \ldots, e-1$,*

$$\frac{\delta_{d_{i_1}+\ldots+d_{i_k}}}{\epsilon_{d_{i_1}+\ldots+d_{i_k}}} = [b_{i_k+1}, \ldots, b_{i_{k+1}-1}]$$

if $i_{k+1} > i_k + 1$, or $\delta_{d_{i_1}+\ldots+d_{i_k}} = 1$ if $i_{k+1} = i_k + 1$. All other δ_i are equal to 0.

In Algorithm 1 it was explained how to obtain the M-resolution corresponding to a given zero continued fraction in $K(\overline{W})$.

Lemma 5.6. *An N-resolution $W^- \to \overline{W}$ associated to the M-resolution $W^+ \to \overline{W}$ can be constructed as follows. It has Wahl singularities $\bar{P}_i$ of type $\frac{1}{\bar{n}_i^2}(1, \bar{n}_i\bar{a}_i - 1)$ for $i = 0, \ldots, r$, which we will describe from the bottom up via $\tilde{n}_p, \tilde{a}_p$ such that $\bar{n}_{r-i} = \tilde{n}_i$, $\bar{a}_{r-i} = \tilde{a}_i$ for $i = 0, \ldots, r$. The algorithm is as follows.*

- *If $i_1 = 1$ (i.e. $d_1 \ne 0$), then $\tilde{n}_p = \tilde{a}_p = 1$ for $p = 0, \ldots, d_1 - 1$. In other words, we start with d_1 smooth points.*
- *If $i_1 > 1$, then $\frac{\tilde{n}_p}{\tilde{n}_p - \tilde{a}_p} = [b_1, \ldots, b_{i_1-1}]$ for $p = 0, \ldots, d_{i_1} - 1$.*
- *Let $q = \sum_{j=1}^{k} d_{i_j}$. Then $\frac{\tilde{n}_p}{\tilde{n}_p - \tilde{a}_p} = [b_1, \ldots, b_{i_{k+1}-1}]$ for $p = q, \ldots, q + d_{i_{k+1}} - 1$.*

The curves $\bar{\Gamma}_i$ for $i = 1, \ldots, r$ are as follows. If $\bar{\Gamma}_i$ passes through one or two Wahl singularities, then its proper transform in the minimal resolution is a (-1)-curve. Otherwise (i.e. no Wahl singularities) it is a (-2)-curve.

Example 5.7. By [KSB, Ex. 3.15], the c.q.s. $\frac{1}{19}(1, 7)$ admits three M-resolutions, where the first M-resolution is the minimal resolution:

$$(3) - (4) - (2), \qquad \left[\binom{2}{1}\right] - (1) - \left[\binom{3}{1}\right], \qquad (3) - \left[\binom{2}{1}\right] - (2).$$

The corresponding N-resolutions are

$$\left[\binom{8}{3}\right]-(1)-\left[\binom{8}{3}\right]-(1)-\left[\binom{2}{1}\right]-(1),\ \left[\binom{5}{2}\right]-(1)-\left[\binom{2}{1}\right],\ \left[\binom{8}{3}\right]-(1)-\left[\binom{5}{2}\right]-(1).$$

Definition 5.8. A Wahl resolution $W \to (P \in \overline{W})$ is called *extremal* if the exceptional divisor consists of a single curve Γ_1. We have two Wahl singularities P_0, P_1 (which may be smooth points). The type of P_i is $\frac{1}{n_i^2}(1, n_i a_i - 1)$ and we have

$$\delta_1 = n_0 n_1 |K_W \cdot \Gamma_1| \quad \text{and} \quad -n_0^2 n_1^2 \Gamma_1^2 = \Delta = n_0^2 + n_1^2 \pm \delta_1 n_0 n_1, \tag{5.1}$$

where $\pm$ is the sign of $K_W \cdot \Gamma_1$. If $\delta_1 = 0$, then we have the M-resolution of $\frac{\Delta}{\Omega} = \frac{2n^2}{2na-1}$ for some $0 < a < n$ coprime [BC2]. If W is an extremal M-resolution with $\delta_1 > 0$, then W is an extremal P-resolution introduced and studied in [HTU].

It is not hard to see that an extremal M-resolution has a unique N-resolution, and this is an initial mk2A when $\delta_1 \neq 0$. The following is the proof taken from [TU, Lemma 2.9].

Lemma 5.9. *An extremal M-resolution has a unique N-resolution.*

Proof. If $\delta_1 = 0$, then we have the M-resolution of $\frac{\Delta}{\Omega} = \frac{2n^2}{2na-1}$ for some $0 < a < n$ coprime. Here N-resolution and M-resolution coincide, and there is only one index i_1 and $d_{i_1} = 2$. If W^+ is an extremal M-resolution with $\delta_1 > 0$, then we have an extremal P-resolution of [HTU]. Here we have only two indices i_1, i_2. We have $d_{i_1} = d_{i_2} = 1$, and

$$[b_1, \ldots, b_{i_1} - 1, \ldots, b_{i_2} - 1 \ldots, b_s] = 0.$$

We now prove that the N-resolution proposed in Lemma 5.6 is indeed an N-resolution. We know that $[b_s, \ldots, b_1] - (1) - [\binom{\bar{n}_1}{\bar{a}_1}] - (1) - [\binom{n_0}{a_0}]$ can be blown-down to $[b_s, \ldots, b_{i_2} - 1, b_{i_2-1}, \ldots, b_1] - (1) - [\binom{n_0}{a_0}]$ and that can be blown-down to $[b_1, \ldots, b_{i_1} - 1, \ldots, b_{i_2} - 1 \ldots, b_s]$, which is zero, and so $\frac{\Delta}{\Omega} = [\binom{\bar{n}_1}{\bar{a}_1}] - (1) - [\binom{n_0}{a_0}]$. Hence, we do get a Wahl resolution $W \to \overline{W}$ in this way.

We now check that $K_W \cdot \Gamma < 0$, where Γ is the central curve, and $\bar{\delta}_1 = \delta_1$. Let $\frac{p_k}{q_k} = [b_1, \ldots, b_{k-1}]$, $p_1 = 1$, $p_0 = q_1 = 0$, and $q_0 = -1$. Then

$$\frac{p_{i_1} q_{i_2} - p_{i_2} q_{i_1}}{p_{i_1} q_{i_2-1} - p_{i_2-1} q_{i_1}} = [b_{i_2-1}, \ldots, b_{i_1+1}] = \frac{\delta_1}{\epsilon_1'}$$

by [HTU, Lemma 4.2]. But, by definition, we have $p_{i_1} = n_0$, $q_{i_1} = n_0 - a_0$, $p_{i_2} = \bar{n}_1$, and $q_{i_2} = \bar{n}_1 - \bar{a}_1$. Therefore, $\delta_1 = \bar{n}_1 a_0 - n_0 \bar{a}_1$. On the other hand, a toric computation shows that $K_W \cdot \Gamma = -1 + \left(1 - \frac{\bar{n}_1 - \bar{a}_1}{\bar{n}_1}\right) + \left(1 - \frac{a_0}{n_0}\right) = -\frac{\bar{n}_1 a_0 - n_0 \bar{a}_1}{\bar{n}_1 n_0}$, and so $K_W \cdot \Gamma$ is negative and $\bar{\delta}_1 = \delta_1$.

Finally, for uniqueness let us consider some Wahl chain $[\binom{\tilde{n}_1}{\tilde{a}_1}]$ such that

$$[b_s, \ldots, b_1] - (1) - \left[\binom{\tilde{n}_1}{\tilde{a}_1}\right] - (1) - \left[\binom{n_0}{a_0}\right] = 0,$$

but then we also have $[\binom{\tilde{n}_1}{\tilde{a}_1}] - (1) - [\binom{n_0}{a_0}] - (1) - [b_s, \ldots, b_1] = 0$, and so $[\binom{\tilde{n}_1}{\tilde{a}_1}]$ is determined, being dual to the contraction of $[\binom{n_0}{a_0}] - (1) - [b_s, \ldots, b_1]$. □

For the proof of Lemma 5.6 see [TU, Lemma 2.6].

Corollary 5.10. *Every M-resolution has a unique associated N-resolution.*

Proof. We know this is true for $r = 1$ (extremal M-resolution). For $r \geq 2$ we go by induction on r. We have that $\Gamma_1 \cup \ldots \cup \Gamma_{r-1}$ is an M-resolution of $\frac{1}{\Delta_{r-1}}(1, \Omega_{r-1})$, and so we can apply induction for all singularities, deltas, and $\frac{1}{\Delta_i}(1, \Omega_i)$ except for $\bar{n}_0, \bar{a}_0$. Let $\frac{\Delta_{r-1}}{\Omega_{r-1}} = [f_1, \ldots, f_t]$. Then we have $[b_s, \ldots, b_1] - (1) - [\binom{n'_0}{a'_0}] - (1) - [f_1, \ldots, f_t] = 0$, and this implies $[\binom{n'_0}{a'_0}] - (1) - [f_1, \ldots, f_t] - (1) - [b_s, \ldots, b_1] = 0$, and so $[\binom{n'_0}{a'_0}]$ is determined by $\frac{1}{\Delta_{r-1}}(1, \Omega_{r-1})$ and $\frac{1}{\Delta}(1, \Omega)$. □

Example 5.11. As we pointed out before, using the computer program MNres [Z1], one can find all M-resolutions and N-resolutions of any c.q.s. For example, let us again consider $\frac{1}{85}(1, 49)$. We have $\frac{85}{49} = [2, 4, 5, 2, 2]$, and $\frac{85}{36} = [3, 2, 3, 2, 2, 4]$. This c.q.s. has a deformation space with 5 irreducible components. For each of them, we list the corresponding: zero continued fraction, dimension of the component, the vector of the δ_i, the M-resolution, and the N-resolution.

$[1, 2, 2, 2, 2, 1]$, dimension is 10, $(0, 2, 3, 0, 0)$ $(2) - (4) - (5) - (2) - (2)$ (minimal resolution) $[\binom{26}{15}] - (1) - [\binom{26}{15}] - (1) - [\binom{26}{15}] - (1) - [\binom{5}{3}] - (1) - (2)$
$[2, 1, 3, 2, 2, 1]$, dimension is 8, $(1, 7, 0, 0)$ $(2) - [\binom{2}{1}] - (5) - (2) - (2)$ $[\binom{26}{15}] - (1) - [\binom{26}{15}] - (1) - [\binom{26}{15}] - (1) - [\binom{3}{2}] - (1)$
$[1, 2, 3, 2, 1, 3]$, dimension is 6, $(0, 8, 1)$ $(2) - (4) - [\binom{3}{1}] - (2)$ $[\binom{26}{15}] - (1) - [\binom{19}{11}] - (1) - (2)$
$[2, 2, 3, 1, 2, 4]$, dimension is 2, (5) $(2) - [\binom{7}{2}]$ (extremal P-resolution) $[\binom{12}{7}] - (1)$
$[3, 1, 3, 2, 1, 4]$, dimension is 2, (5) $[\binom{3}{2}] - (1) - [\binom{4}{1}]$ (extremal P-resolution) $[\binom{19}{11}] - (1) - [\binom{3}{2}]$

We finish showing the N-resolution corresponding to the minimal resolution. In general, we write the Hirzebruch-Jung continued fraction as

$$\frac{\Delta}{\Omega} = [\underbrace{2, \ldots, 2}_{y_1}, x_1, \underbrace{2, \ldots, 2}_{y_2}, x_2, \ldots, \underbrace{2, \ldots, 2}_{y_{e-1}}, x_{e-1}, \underbrace{2, \ldots, 2}_{y_e}],$$

where $y_i \geq 0$ and $x_i \geq 3$ for all i. This describes the minimal resolution W^+. We now compute the N-resolution W^- of W^+ explicitly. The dual fraction is

$$\frac{\Delta}{\Delta - \Omega} = [y_1 + 2, \underbrace{2, \ldots, 2}_{x_1 - 3}, y_2 + 3, \underbrace{2, \ldots, 2}_{x_2 - 3}, y_3 + 3, \ldots, y_{e-1} + 3, \underbrace{2, \ldots, 2}_{x_{e-1} - 3}, y_e + 2].$$

We have that the $d_i \neq 0$ are exactly in the positions of the y_i. In particular, if $d_{i_1}, \ldots, d_{i_e}$ are the $d_i \neq 0$ with $i_1 < \ldots < i_e$, then $i_1 = 1$, i_e is the index of the last position, and $d_{i_k} = y_k + 1$ for all k. Note that the non zero $\bar{\delta}$ are computed via $[\underbrace{2, \ldots, 2}_{x_i - 3}]$, and so they are equal to $x_i - 2$. We have that the data $\bar{n}_{i_k}, \bar{a}_{i_k}$ for the distinct Wahl singularities in the N-resolution is

$$\frac{\bar{n}_{i_k}}{\bar{n}_{i_k} - \bar{a}_{i_k}} = [y_1 + 2, \underbrace{2, \ldots, 2}_{x_1 - 3}, y_2 + 3, \ldots, y_{k-2} + 3, \underbrace{2, \ldots, 2}_{x_{k-2} - 3}, y_{k-1} + 3, \underbrace{2, \ldots, 2}_{x_{k-1} - 3}]$$

for $k > 1$, and smooth point for $k = 1$.

Note that we have

$$\Delta = \bar{n}_e(y_e + 1) + \bar{n}_{e-1}(y_{e-1} + 1) + \ldots + \bar{n}_2(y_2 + 1) + (y_1 + 1). \tag{5.2}$$

Indeed, let us consider the matrix

$$M = \begin{bmatrix} y_e + 2 & -1 & & & & & & \\ -1 & 2 & -1 & & & & & \\ & & \ddots & & & & & \\ & -1 & 2 & -1 & & & & \\ & & -1 & y_{e-1} + 3 & -1 & & & \\ & & & & \ddots & & & \\ & & & & -1 & 2 & -1 \\ & & & & & -1 & y_1 + 2 \end{bmatrix},$$

whose diagonal has the sequence

$$\{y_e + 2, \underbrace{2, \ldots, 2}_{x_{e-1} - 3}, y_{e-1} + 3, \ldots, y_3 + 3, \underbrace{2, \ldots, 2}_{x_2 - 3}, y_2 + 3, \underbrace{2, \ldots, 2}_{x_1 - 3}, y_1 + 2\}.$$

Its determinant is equal to Δ. On the other hand, we can use the linearity of the determinant on its first row $(y_e + 2, -1, 0, \ldots, 0) = (1, -1, 0, \ldots, 0) + (y_e + 1, 0, 0, \ldots, 0)$, via the sum $M = M_1 + M_2$ where M_1 corresponds to the continued fraction

$$[1, \underbrace{2, \ldots, 2}_{x_{e-1} - 3}, y_{e-1} + 3, \ldots, y_3 + 3, \underbrace{2, \ldots, 2}_{x_2 - 3}, y_2 + 3, \underbrace{2, \ldots, 2}_{x_1 - 3}, y_1 + 2],$$

and $\det(M_2) = (y_e + 1)\bar{n}_e$. But then $\det(M_1)$ is the numerator of the continued fraction

$$[y_{e-1} + 2, \ldots, y_3 + 3, \underbrace{2, \ldots, 2}_{x_2 - 3}, y_2 + 3, \underbrace{2, \ldots, 2}_{x_1 - 3}, y_1 + 2],$$

by contracting the 1 and the consecutive 2s in the diagonal of M_1. Now we use induction on e to write the claimed formula.

In the next section, we will see how to obtain the unique N-resolution from a given M-resolution using a suitable action of the braid group, which is defined via antiflipping extremal M-resolutions.

Exercises.

(1) Verify why in Definition 5.3 is necessary to have (1), (2) and (3) for uniqueness.

(2) ⋆ Let $n_0, \ldots, n_r$ be the indices for an M-resolution, and $\bar{n}_0, \ldots, \bar{n}_r$ be the indices for its N-resolution, all over the c.q.s. $\frac{1}{\Delta}(1, \Omega)$. Show that

$$\Delta = n_0 \bar{n}_r + n_1 \bar{n}_{r-1} + \ldots + n_r \bar{n}_0.$$

One instance of this was shown in Equation 5.2 for the minimal resolution.

5.2. **Braid group action.**

Given a c.q.s. $\overline{W}$, we will show how to connect an M-resolution W^+ with its N-resolution W^- by a sequence of antiflips, which are generators of the braid group B_{r+1} action on the set of Wahl resolutions $W \to \overline{W}$ with $r+1$ Wahl singularities. We recall that the braid group on $r+1$ strands can be presented as

$$B_{r+1} = \langle \theta_1, \ldots, \theta_r \mid \theta_i \theta_{i+1} \theta_i = \theta_{i+1} \theta_i \theta_{i+1}, \ \theta_i \theta_j = \theta_j \theta_i \rangle,$$

where in the first group of relations $1 \leq i \leq r-1$ and in the second $|i-j| \geq 2$. For example $B_2 = \mathbb{Z}$. This group is torsion free, and for $r \geq 2$, B_{r+1} contains the free group in two generators.

We first describe the action of B_2 on extremal Wahl resolutions $W \to \overline{W}$, where either $K_W \cdot \Gamma_1 > 0$ (extremal P-resolutions), $K_W \cdot \Gamma_1 < 0$ (*K-negative resolutions*), or $K_W \cdot \Gamma_1 = 0$ when $\delta_1 = 0$ (*K-trivial resolutions*). We will refer to the action of a generator of B_2 as the *right antiflip* and to its inverse as the *left antiflip*.

A $\mathbb{Q}$-Gorenstein smoothing $(W \subset \mathcal{W}) \to (0 \in \mathbb{D})$ of an extremal Wahl resolution W over a smooth curve $\mathbb{D}$ can be blown-down to a smoothing $\overline{\mathcal{W}} \to \mathbb{D}$ of $\overline{W}$. This gives a threefold contraction $\mathcal{W} \to \overline{\mathcal{W}}$, which is $K_{\mathcal{W}}$-positive, $K_{\mathcal{W}}$-negative, or $K_{\mathcal{W}}$-trivial depending on the three cases above. The antiflip is defined differently in each case.

Antiflips: K-positive case. Consider a $\mathbb{Q}$-Gorenstein smoothing $\mathcal{W}^+ \to \mathbb{D}$ of an extremal P-resolution W^+ over a smooth curve. This situation may not have an anti-flip of type mk1Aor mk2A, this depends on the "direction" of the deformation. This was studied in detail in [HTU]. So, when we have the correct direction, then we may anti-flip $\mathcal{W}^+ \to \mathbb{D}$. We will only consider the "initial" mk2Aextremal neighborhoods as antiflips. To differentiate them, we use left L and right R as indicated in Figure 21. We refer to $W_0^- \to \overline{W}$ as the *right antiflip* (or just the antiflip) of an extremal P-resolution $W^+ \to \overline{W}$ and to $W_1^- \to \overline{W}$ as the *left antiflip*.

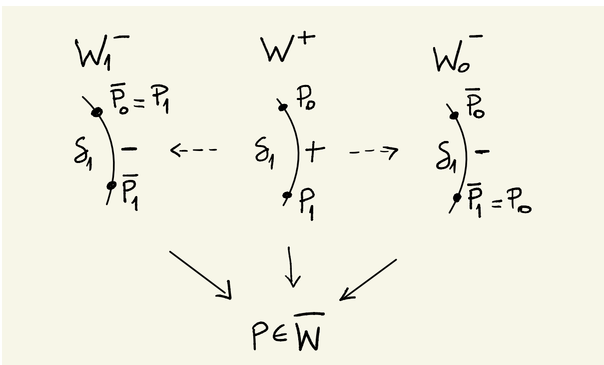


FIGURE 21. Left and right antiflips of $\mathcal{W}^+ \to \mathbb{D}$

Antiflips: K-negative case. This is what we do in a Mori train going from one mk2Ato the next mk2A. If we move keeping P_0 as shown in Figure 21 (changing W^+ by a W with $\Gamma_1 \cdot K_W < 0$), then this is the right anti-flip R , otherwise we have the left L.

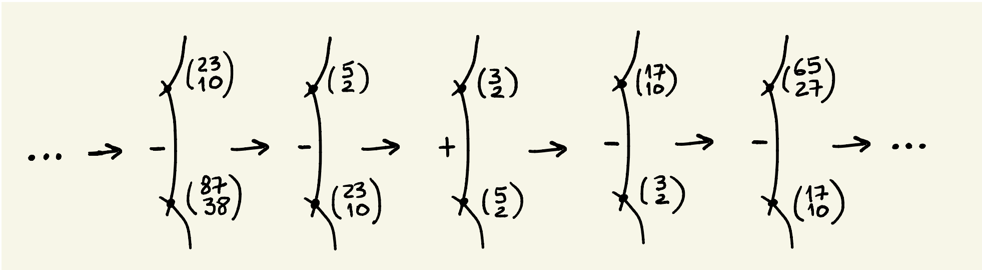


FIGURE 22. Anti-flips in the Mori trains of a given W^+.

Antiflips: K-trivial case. Here W^+ is $[\binom{n}{a}]-(1)-[\binom{n}{a}]$ (unless $n=1$ in which case it is a (-2)-curve with two smooth points) and $\overline{W} = \frac{1}{2n^2}(1, 2na-1)$ [BC2]. The blow-down $\overline{\mathcal{W}}$ of a $\mathbb{Q}$-Gorenstein smoothing $\mathcal{W}^+$ of W^+ is a $\mathbb{Q}$-Gorenstein smoothing of $\overline{W}$. The contraction $\mathcal{W}^+ \to \overline{\mathcal{W}}$ is crepant, and it can be flopped giving a threefold $\mathcal{W}'$ (which we call an antiflip of $\mathcal{W}^+$) with a central fiber $W' \simeq W^+$ (see [BC2] or [K4, Section 5]), which we call an antiflip of W^+ in the K-trivial case. This is the Atiyah flop when we have no Wahl singularities.

The following will be assumptions that we will impose on W so that we have plenty of freedom on its deformations, and we have the "starting" point in the s.o.d.s of the next sections. Sometimes we can relax some of the hypothesis to obtain results on more general surfaces. We will try to indicate when that happens.

Assumption 5.12.
(1) W is a normal projective surface with Wahl singularities $\{P_0, \ldots, P_r\}$, smooth outside of them. We have a Wahl resolution $\Gamma_1 \cup \ldots \cup \Gamma_r \subset W$ of $P \in \overline{W}$, and so the contraction of the chain $\Gamma_1, \ldots, \Gamma_r$ to the point $P \in \overline{W}$.
(2) $q(W) = \dim H^1(W, \mathcal{O}_W) = 0$ and $p_g(W) = \dim H^2(W, \mathcal{O}_W) = 0$.
(3) There is a Weil divisor $\bar{A}$ on $\overline{W}$ that generates the local class group $\mathrm{Cl}(P \in \overline{W})$. By Lemma 5.20 (at the end of this section) we can choose effective smooth divisors $\bar{A}, \tilde{\bar{A}} \subset \overline{W}$ such that the germ $P \in (\bar{A} \cup \tilde{\bar{A}}) \subset \overline{W}$ is etále-locally isomorphic to $0 \in (x=0) \cup (y=0) \subset \frac{1}{\Delta}(1, \Omega)$. Proper transforms Γ_0 (resp. Γ_{r+1}) of $\bar{A}$ (resp. $\tilde{\bar{A}}$) in the Wahl resolution W of $\overline{W}$ intersect the chain $\Gamma_1 \cup \ldots \cup \Gamma_r$ only at P_0 (resp. P_r), where they are equivalent to toric boundaries opposite to Γ_1 (resp. Γ_r) as in Figure 20.
(4) $H^2(\overline{W}, T_{\overline{W}}) = 0$. By Lemma 5.21, there are no local-to-global obstructions to $\mathbb{Q}$-Gorenstein deformations of a Wahl resolution W of $\overline{W}$ or the pair (W, Δ) where $\Delta = \Gamma_0 + \Gamma_1 + \ldots + \Gamma_r + \Gamma_{r+1}$ if (2) and (3) also hold. A general example satisfying (4) is any $\overline{W}$ with $-K_{\overline{W}}$ big [HP, Prop. 3.1].

We note that even rational surfaces may not satisfy Assumption 5.12 (3), see e.g. [KKS, Examples 5.4 and 5.5]. On the other hand, rational surfaces with big K_W may not satisfy Assumption 5.12 (4), see e.g. [RU3, Sections 4 and 5].

We will define the action of generators of B_{r+1} on Wahl resolutions with $r+1$ singularities by treating every irreducible curve in its exceptional divisor as an extremal Wahl resolution. Relations of the braid group are checked in Theorem 5.18.

Definition 5.13. Let $W \to \overline{W}$ be a Wahl resolution with exceptional divisor $\Gamma_1 \cup \ldots \cup \Gamma_r$ and toric boundaries Γ_0 and Γ_{r+1} as in Lemmas 5.20 and 5.21. The neighborhood of $\Gamma_i \subset W$ contains a subchain $[\binom{n_{i-1}}{a_{i-1}}] - (c_i) - [\binom{n_i}{a_i}]$ of an extremal Wahl resolution. We have a contraction $W \to W_i$ of $\Gamma_i \subset W$ into a c.q.s. $\frac{1}{\Delta_{\Gamma_i}}(1, \Omega_{\Gamma_i})$, which has as toric boundary the image of Γ_{i-1} and Γ_{i+1}. By Lemma 5.21, we can choose two deformations of W_i (the same ones if the extremal resolution is a P-resolution or a K-trivial resolution) which (1) are equisingular at singularities of W_i other than $\frac{1}{\Delta_{\Gamma_i}}(1, \Omega_{\Gamma_i})$, (2) lift the boundary of W_i, and (3) smoothen $\frac{1}{\Delta_{\Gamma_i}}(1, \Omega_{\Gamma_i})$ as in the discussion of antiflips of extremal Wahl resolutions in the beginning of this section. These deformations of W_i are blow-downs of $\mathbb{Q}$-Gorenstein deformations of W and another Wahl resolution $R_i(W) \to \overline{W}$, respectively. We call $R_i(W)$ the *right antiflip* of $W \to \overline{W}$ at Γ_i. The *left antiflip* is defined is a similar way. The singularities of $R_i(W)$ and $L_i(W)$ are the same as for W except at the positions $i-1$ and i, where we have the singularities produced by the antiflip of an extremal Wahl resolution $[\binom{n_{i-1}}{a_{i-1}}] - (c_i) - [\binom{n_i}{a_i}]$.

The following is [TU, Corollary 3.4].

Corollary 5.14. *Given a sequence of Wahl resolutions $W_0, W_1, \ldots, W_k \to \overline{W}$ with Wahl chains $\Gamma_0^j, \ldots, \Gamma_{r+1}^j$ for $j = 0, \ldots, k$, suppose $W_i = R_{l_i}(W_{i-1})$ for $i = 1, \ldots, k$.*

(1) There is a sequence of $\mathbb{Q}$-Gorenstein smoothings $Y_i \rightsquigarrow W_i$ for $i = 0, 1, \ldots, k$ over smooth curve germs $\mathbb{D}_i$ that belong to the same component of $\mathrm{Def}_{P \in \overline{W}}$.

(2) If $K_{W_{i-1}} \cdot \Gamma_{l_i}^{i-1} \geq 0$ for $i = 1, \ldots, k$, i.e. on every step we antiflip an extremal P-resolution or a K-trivial resolution, then we can assume that $\mathbb{D}_1 = \ldots = \mathbb{D}_k = \mathbb{D}$ is the same curve in $\mathrm{Def}_{P \in \overline{W}}$ and $(W_{i-1} \subset \mathcal{W}_{i-1}) \to (0 \in \mathbb{D})$ is the flip (or flop) of $(W_i \subset \mathcal{W}_i) \to (0 \in \mathbb{D})$ for all $i = 1, \ldots, k$ with respect to the contraction of $\Gamma_{l_i}^i \subset W_i$. In particular, the smooth fibers Y_i of these families are isomorphic.

Proof. (1) is clear. To prove (2), choose a $\mathbb{Q}$-Gorenstein smoothing $(W_k \subset \mathcal{W}_k) \to (0 \in \mathbb{D})$ over a smooth curve germ $\mathbb{D}$ with all axial multiplicities equal to 1, which exists by Lemma 5.21. Then we apply a sequence of flips (or flops if $\delta_{l_i}^i = 0$) to contractions of $\Gamma_{l_i}^i \subset \mathcal{W}_i$ for $i = k, k-1, \ldots, 1$. □

Proposition 5.15. *[[TU, Proposition 3.5]] Let $W \to \overline{W}$ be a Wahl resolution with a chain of 3 curves $\Gamma_1, \Gamma_2, \Gamma_3$, and singularities P_0, P_1, P_2, P_3 where the type of P_i is $\frac{1}{n_i^2}(1, n_i a_i - 1)$. Consider $W' := R_2(W)$ the right antiflip of $W \to \overline{W}$ at Γ_2. Hence we have a Wahl resolution $W' \to \overline{W}$ with a chain of 3 curves $\Gamma_1', \Gamma_2', \Gamma_3'$, and singularities $P_0' = P_0, P_1', P_2' = P_1, P_3' = P_3$. Let $\delta_i' = n_{i-1}' n_i' |K_{W'} \cdot \Gamma_i'|$. Then we have the following three situations:*

***(-/-)**: $\Gamma_2 \cdot K_W < 0$ and $\Gamma_2' \cdot K_{W'} < 0$.*

(1) $n_1' = \delta_2 n_1 - n_2$, $a_1' = \delta_2 a_1 - a_2$, $n_2' = n_1$, $a_2' = a_1$, $\delta_2' = \delta_2$.

(2) $n_0' n_1' \, \Gamma_1' \cdot K_{W'} = \frac{\pm\delta_1(\delta_2 n_1 - n_2) + \delta_2 n_0}{n_1}$, where $\pm$ is the sign of $K_W \cdot \Gamma_1$.

(3) $n_2' n_3' \, \Gamma_3' \cdot K_{W'} = \frac{\pm\delta_3 n_1 - \delta_2 n_3}{n_2}$, where $\pm$ is the sign of $K_W \cdot \Gamma_3$.

***(-/+)**: $\Gamma_2 \cdot K_W < 0$ and $\Gamma_2' \cdot K_{W'} > 0$. Let $-c_2'$ be the self-intersection of the proper transform of Γ_2' in the minimal resolution of W'.*

(1) $n'_1 = n_2 - \delta_2 n_1$, $a'_1 = a_2 - \delta_2 a_1 - (c'_2 - 1)n'_1$, $n'_2 = n_1$, $a'_2 = a_1$, $\delta'_2 = \delta_2$, *and* $\delta'_2 = (c'_2 - 1)n'_1 n_1 + n_1 a'_1 - n'_1 a_1$.
(2) $n'_0 n'_1 \, \Gamma'_1 \cdot K_{W'} = \frac{\pm\delta_1(n_2 - \delta_2 n_1) - \delta_2 n_0}{n_1}$*, where* $\pm$ *is the sign of* $K_W \cdot \Gamma_1$*.*
(3) $n'_2 n'_3 \, \Gamma'_3 \cdot K_{W'} = \frac{\pm\delta_3 n_1 - \delta_2 n_3}{n_2}$*, where* $\pm$ *is the sign of* $K_W \cdot \Gamma_3$*.*

(+/-): $\Gamma_2 \cdot K_W \geq 0$ *and* $\Gamma'_2 \cdot K_{W'} \leq 0$*. Let* $-c_2$ *be the self-intersection of the proper transform of* Γ_2 *in the minimal resolution of* W*.*
(1) $n'_1 = \delta_2 n_1 + n_2$, $a'_1 = \delta_2 a_1 + a_2 - (c_2 - 1)n_2$, $n'_2 = n_1$, $a'_2 = a_1$, $\delta'_2 = \delta_2$.
(2) $n'_0 n'_1 \, \Gamma'_1 \cdot K_{W'} = \frac{\pm\delta_1(\delta_2 n_1 + n_2) + \delta_2 n_0}{n_1}$*, where* $\pm$ *is the sign of* $K_W \cdot \Gamma_1$*.*
(3) $n'_2 n'_3 \, \Gamma'_3 \cdot K_{W'} = \frac{\pm\delta_3 n_1 + \delta_2 n_3}{n_2}$*, where* $\pm$ *is the sign of* $K_W \cdot \Gamma_3$*.*

In particular, we have in all cases that

$$K_W \cdot \Gamma_1 = K_{W'} \cdot (\Gamma'_1 + \Gamma'_2) \quad \textit{and} \quad K_{W'} \cdot \Gamma'_3 = K_W \cdot (\Gamma_2 + \Gamma_3).$$

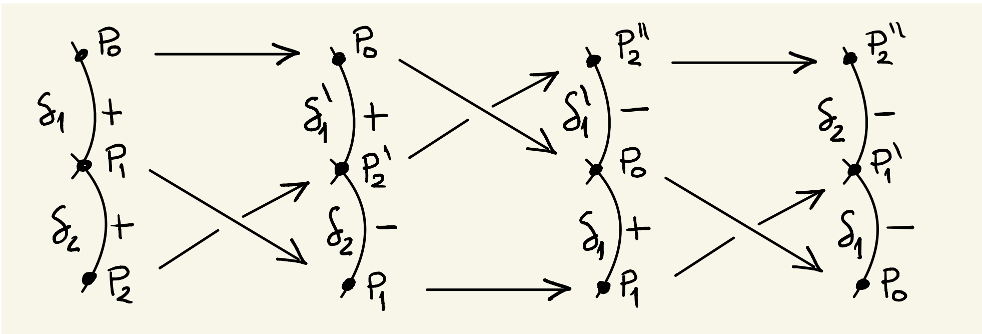


FIGURE 23. Proposition 5.15 in a picture.

A careful computation shows the following.

Lemma 5.16. *Let* $W_0 \to \overline{W}$ *be a Wahl resolution. Then we have the braid relation*

$$R_2 R_1 R_2(W_0) = R_1 R_2 R_1(W_0).$$

Theorem 5.17. *[*[TU, Theorem 3.8]*] After applying* $r(r+1)/2$ *right antiflips of curves contained in the Wahl resolutions starting with* $W^+ \to \overline{W}$*, we get the corresponding N-resolution* $W^- \to \overline{W}$*. On every step, we antiflip either an extremal P-resolution or a curve with* $\delta = 0$*.*

Theorem 5.18. *[*[TU, Theorem 3.9]*] The operations of right antiflips* R_i *on Wahl resolutions* $W \to \overline{W}$ *with* $r+1$ *singularities satisfy braid relations* $R_i R_j = R_j R_i$ *for* $i > j+1$ *and* $R_i R_{i+1} R_i = R_{i+1} R_i R_{i+1}$*. In particular, they give the action of the braid group* B_{r+1}*.*

Corollary 5.19. *Every Wahl resolution* $W \to \overline{W}$ *is in the braid group orbit of a unique M-resolution* $W^+ \to \overline{W}$*.*

Visualization problem: Given an M-resolution $W^+ \to \overline{W}$ with $r = 2$, we suppose to have a 3-dimensional deformation space where deformations of (W^+, B). Inside here, we would have all deformations coming from all Wahl resolutions $W \to \overline{W}$. What is literally the picture? This would be the analog of the universal antiflip of extremal P-resolutions as in Figure 24.

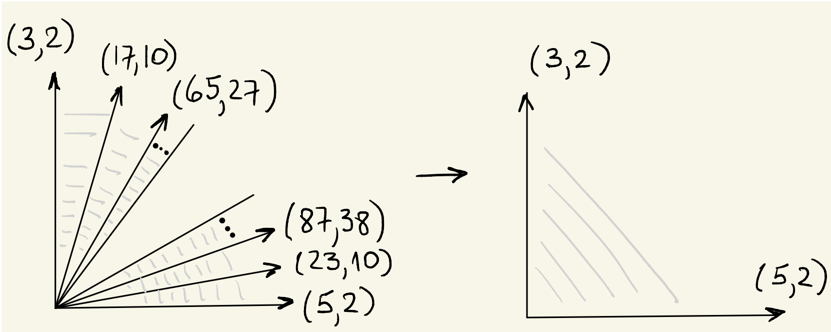


FIGURE 24. Universal family of antiflips (on the left) of the extremal P-resolution (on the right) for [HTU, Figure 1].

Lemma 5.20. *[[TU, Lemma 3.1]] Let $P \in \overline{W}$ be a c.q.s. surface satisfying Assumption 5.12 (1), (3). We can choose effective, smooth divisors $\bar{A}$ and $\tilde{\bar{A}}$ such that the germ $P \in (\bar{A} \cup \tilde{\bar{A}}) \subset \overline{W}$ is etále-locally isomorphic to the germ $0 \in (x = 0) \cup (y = 0) \subset \mathbb{C}^2/\mu_\Delta$. Proper transforms Γ_0 of $\bar{A}$ and Γ_{r+1} of $\tilde{\bar{A}}$ in any Wahl resolution W of $\overline{W}$ intersect the chain $\Gamma_1 \cup \ldots \cup \Gamma_r$ only at the end-points P_0 and P_r, where they give toric boundaries opposite to Γ_1 (resp. Γ_r).*

Lemma 5.21. *[[TU, Lemma 3.2]] Let $\pi\colon W \to \overline{W}$ be a Wahl resolution satisfying Assumption 5.12.*

(1) We can choose divisors Γ_0 and Γ_{r+1} as in Lemma 5.20 so that there are no local-to-global obstructions to deformations of a pair (W, B), where B is the boundary $\Gamma_0 + \Gamma_1 + \ldots + \Gamma_r + \Gamma_{r+1}$, i.e. the morphism $\mathrm{Def}_{(W,B)} \to \prod_{P_i \in W} \mathrm{Def}_{P_i \in (W,B)}$ is smooth.

(2) If W is a Wahl resolution, then there are no local-to-global obstructions to $\mathbb{Q}$-Gorenstein deformations of W or (W, B), for example, there exists a $\mathbb{Q}$-Gorenstein smoothing $Y \rightsquigarrow W$ with a lifting of B for any choice of axial multiplicities $\alpha_0, \ldots, \alpha_r$.

Exercises.

(1) Typically in the examples of W-surfaces in the previous sections, one can find M-resolutions of c.q.s. That is key to construct in the next section exceptional collections of vector bundles. Find some examples with embedded M-resolutions (not only extremal P-resolutions).

(2) $\star$ Find a representation of the dynamical orbits of the Braid group action on an M-resolution. For example, when the M-resolution is an extremal P-resolution, then this is represented by the Mori trains. That is $r = 1$. What is the situation for $r = 2$? Any r?

6. EXCEPTIONAL COLLECTIONS OF HACKING BUNDLES

The aim of this chapter is to show how to construct exceptional vector bundles on the nonsingular fibers W_t of a W-surface $W_t \rightsquigarrow W$, under certain hypotheses on the singular surface W. This is the work of Hacking [H4], we will call them *Hacking bundles*. Depending on the particular geometry of W, one can actually construct exceptional collections of these Hacking bundles on W_t. We recall that an exceptional collection defines a *semi-orthogonal decomposition* of the derived category $D^b(W_t)$. This is the main motivation, along with the desire to understand the orthogonal complement of the exceptional collection in $D^b(W_t)$. Particularly interesting are long exceptional collections, and W-surfaces provide plenty of them.

Derived categories of algebraic varieties is a central area in algebraic geometry; see the ICM talks by Bondal-Orlov [BO], and Kuznetsov [K2], [K3]. In particular, semi-orthogonal decompositions and their behavior under deformations have recently attracted quite a lot of attention; see for example [K1], [K3], [K4], [TU]. The works of Kalck–Karmazyn [KK1], Kawamata [K1], Karmazyn–Kuznetsov–Shinder [KKS] develop a way to express the derived categories of singular surfaces with a c.q.s. $\frac{1}{\Delta}(1,\Omega)$. That was the starting point of [TU], which will be explained by the end of this chapter.

One main result from [TU] was the discovery of N-resolutions, which were explained in the previous chapter, and their application to mutations of Hacking exceptional collections. For example, under certain hypotheses, we can find on many W-surfaces long exceptional collections that are strong. For this, it is key the use of the birational geometry of W-surfaces.

6.1. Hacking exceptional bundles.

Definition 6.1. An exceptional vector bundle (e.v.b.) E on a projective surface Y is a locally free sheaf such that $\mathrm{Hom}(E,E)=\mathbb{C}$, and $\mathrm{Ext}^1(E,E)=\mathrm{Ext}^2(E,E)=0$.

In this way, an e.v.b. is indecomposable, rigid (no infinitesimal deformations), and unobstructed in deformations. Given an e.v.b. E we have that its dual $E^\vee$ and $E\otimes\mathcal{L}$ are e.v.b.s, where $\mathcal{L}$ is a line bundle.

An exceptional vector bundle E on Y induces the semi-orthogonal decomposition (s.o.d.) $D^b(Y)=\langle E^\perp, E\rangle$, where E is the admissible triangulated subcategory generated by E and $E^\perp=\{t\in D^b(Y)\colon \ \mathrm{Hom}(e,t)=0,\ \forall\, e\in E\}$; see [H10, §1.4]. At this point, we should mention that conjecturally there is a strong restriction on the existence of an e.v.b. (Folklore conjecture? see [BGL]):

Conjecture 6.2. *Let Y be a nonsingular projective variety. If the canonical bundle K_Y is nef and $h^0(Y,K_Y)>0$, then X admits no non-trivial semi-orthogonal decomposition.*

Since our W-surfaces $W_t \rightsquigarrow W$ with W singular and K_W nef have $q=0$, we should then consider only $p_g=q=0$ surfaces. Although it would be instructive to keep in mind that a W-surface $W_t \rightsquigarrow W$ may have $p_g>0$; see many examples in [RU2] for $p_g=1$. Also, we note that any line bundle on a $p_g=q=0$ surface is exceptional.

Theorem 6.3 (Theorem 1.1 [H4])**.** *Let $(W\subset\mathcal{W})\to(0\in\mathbb{D})$ be a W-surface such that*

(1) W has only one Wahl singularity $\frac{1}{n^2}(1,na-1)$, $p_g(W)=q(W)=0$, and

(2) the induced exact sequence

$$0\to \mathrm{Pic}(W_t)=H_2(W_t)\to \mathrm{Cl}(W)=H_2(W)\to \mathbb{Z}/n\to 0$$

is exact (see Subsection 4.1), where $t\neq 0$.

Then, possibly after a base change $\mathbb{D} \to \mathbb{D}$, there exists a reflexive sheaf $\mathcal{E}$ (i.e. $\mathcal{E} \to \mathcal{E}^{\vee\vee}$ is an isomorphism) on $\mathcal{W}$ such that

(a) $E := \mathcal{E}|_{W_t}$ *is an e.v.b. of rank* n*, and*

(b) $E_0 := \mathcal{E}|_W$ *is a torsion-free sheaf on* W *such that* $E_0^{\vee\vee}$ *(reflexive hull) is isomorphic to the direct sum of* n *copies of a reflexive rank* 1 *sheaf* A*, and the quotient* $E_0^{\vee\vee}/E_0$ *is a torsion sheaf supported at* $\frac{1}{n^2}(1, na-1)$.

If $\mathcal{H}$ is a line bundle on $\mathcal{W}$ which is ample on the fibers, then E is slope stable with respect to $\mathcal{H}|_{W_t}$. Moreover we have

$$c_1(E) = nc_1(A) \qquad c_2(E) = \frac{n-1}{2n}\big(c_1^2(E) + n + 1\big) \qquad c_1(E) \cdot K_{W_t} \equiv \pm a (mod\ n).$$

There is a slight generalization of this theorem (and different construction) due to Kawamata in [K4, Theorem 1.1]. A *Hacking vector bundle* on a nonsingular projective surface Y is an e.v.b. isomorphic to some e.v.b. E as in Theorem 6.3. To construct them, we need a W-surface with $p_g = q = 0$ and one Wahl singularity, and the assumption (2) in Theorem 6.3. As we saw in Subsection 4.1, this assumption is satisfied, for example, when $H_1(W_t) = 0$.

The construction of Hacking goes roughly as follows. Given $(W \subset \mathcal{W}) \to (0 \in \mathbb{D})$, after a suitable base change and weighted blow-up, he considers a $\mathbb{Q}$-Gorenstein smoothing

$$(W' \cup Z \subset \mathcal{Z}) \to (0 \in \mathbb{D}),$$

where the singular fiber has two components W' and Z. The induced birational morphism $W' \to W$ is a partial resolution over the Wahl singularity extracting the first curve from its minimal resolution. The surface Z is $(w^n + t^a = uv) \subset \mathbb{P}(1, na-1, a, n)$. The intersection $(t = 0) = W' \cap Z$ is a $\mathbb{P}^1$, and it is a simple normal intersection except at the unique c.q.s. in both surfaces (this is called an *orbifold normal crossings* singularity). Inductively on the index n of the Wahl singularity, he produces an e.v.b. on Z of rank n which glues along $W' \cup Z$ to A^n on W', for some divisor A, which exists because of hypothesis (b) in Theorem 6.3. The inductive construction uses an explicit further degeneration of Z into $\mathbb{P}(1, na-1, a^2)$. It is just a $\mathbb{Q}$-Gorenstein smoothing of $\frac{1}{a^2}(1, an-1)$ keeping the other c.q.s. From $\mathbb{P}(1, na-1, a^2)$ we have, by induction, an exceptional pair $(\mathcal{O}, F_1)$ where F_1 is e.v.b. of rank $a < n$ on Z. Its mutation $(F_2^\vee, \mathcal{O})$ produces the e.v.b. F_2 of rank n to do the construction. All details are explained in [H4].

If E is the e.v.b. in Theorem 6.3, then by Riemann-Roch we have

$$2n\big(h^0(E) - h^1(E) + h^2(E)\big) = n^2(A^2 - A \cdot K_W + 1) + 1.$$

We recall that A^2 and $A \cdot K_W$ are rational.

Theorem 6.4 (Theorem 6.4 [H4], Proposition 3.15 [A1])**.** *Let $\mathcal{S}$ be the set of isomorphism classes of normal surfaces W with one Wahl singularity in a W-surface $\mathbb{P}^2 \rightsquigarrow W$. Let $\mathcal{T}$ be the set of isomorphism classes of e.v.b. E on $\mathbb{P}^2$ of rank > 1 modulo $E \mapsto E^\vee$ and $E \mapsto E \otimes \mathcal{L}$ with $\mathcal{L} \in \mathrm{Pic}(\mathbb{P}^2)$.*

Then Theorem 6.3 defines a bijection $\Phi \colon \mathcal{S} \to \mathcal{T}$, $W \mapsto E$.

The bijection in Theorem 6.4 is very particular of $\mathbb{P}^2$, but the potential surjection of $\Phi \colon \mathcal{S} \to \mathcal{T}$ for other surfaces W_t is very interesting. Hacking vector bundles suppose to be a replacement for the nonexistent vanishing cycles in $\mathbb{Q}$-Gorenstein smoothings.

Exercises.

(1) Let W be the contraction of the (-4)-curve in the Hirzebruch surface $\mathbb{F}_4$. We know the existence of a W-surface $\mathbb{P}^2 \rightsquigarrow W$. Let $A = \mathbb{P}^1$ be the image of a fiber of $\mathbb{F}_4 \to \mathbb{P}^1$ in W. Show that Theorem 6.3 works and with this A, obtaining an e.v.b. E of rank 2 on $\mathbb{P}^2$. Show that the slope of E with respect to $\mathcal{O}(1)$ is $\frac{1}{2}$. As in Theorem 6.4, one can prove that every rank 2 e.v.b. on $\mathbb{P}^2$ is of the form $E \otimes \mathcal{O}(\ell)$ (or $E^\vee \otimes \mathcal{O}(\ell)$, but in this case is not necessary) for some $\ell \in \mathbb{Z}$.

(2) $\star\star$ One may think that the previous exercise extends for all e.v.b. on $\mathbb{P}^2$. But Markov uniqueness conjecture is present here in the equivalent form: Up to dualizing and tensoring by line bundles, an e.v.b. on $\mathbb{P}^2$ is uniquely determined by its rank (conjectured by A. N. Tyurin [R3]).

6.2. Hacking exceptional collections.

Definition 6.5. An *exceptional collection of vector bundles* (e.c.) on a projective surface Y is a collection of e.v.b. $E_r, E_{r-1}, \ldots, E_0$ such that

$$\operatorname{Ext}^k(E_i, E_j) = 0$$

for all $i < j$ and all $k \geq 0$. Its *length* is $r+1$. It is said to be *strong* if moreover $\operatorname{Ext}^k(E_i, E_j) = 0$ for all $i > j$, and all $k > 0$.

Proposition 6.6. *If $E_r, E_{r-1}, \ldots, E_0$ is an e.c. on a nonsingular Y, then*

- *$E_r \otimes \mathcal{L}, E_{r-1} \otimes \mathcal{L}, \ldots, E_0 \otimes \mathcal{L}$ is an e.c. for any line bundle $\mathcal{L}$.*
- *$E_0^\vee, E_1^\vee, \ldots, E_r^\vee$ is an e.c. where $E^\vee$ is the dual of E.*
- *$E_0 \otimes \mathcal{O}_Y(K_Y), E_r, E_{r-1}, \ldots, E_1$ is an e.c. where K_Y is a canonical divisor.*
- *$E_{r-1}, \ldots, E_1, E_0, E_r \otimes \mathcal{O}_Y(-K_Y)$ is an e.c. where K_Y is a canonical divisor.*

As explained in [H10, §1.4], an exceptional collection $E_r, E_{r-1}, \ldots, E_0$ on Y defines a s.o.d. $D^b(Y) = \langle \mathcal{A}^\perp, E_r, E_{r-1}, \ldots, E_0, \rangle$ where $\mathcal{A}$ is generated by $E_r, E_{r-1}, \ldots, E_0$. The exceptional collection is *full* if $\mathcal{A}^\perp = \emptyset$. There is the following conjecture by Orlov.

Conjecture 6.7. *The only nonsingular projective varieties that admit a full exceptional collection are rational.*

For example, any exceptional collection on a del Pezzo surface can be completed into a full exceptional collection [KO]. Hence in a del Pezzo surface a maximal length exceptional collection is full. On the other hand, there are rational surfaces that admit an exceptional collection of maximal length but it is not full [K3]. *What is the maximal possible length for a given surface?* We review that now, and later we will define the maximum length.

Let Y be a nonsingular surface. An s.o.d. produces a decomposition of $K_0(Y) = K_0(D^b(Y))$, the Grothendieck group of Y [KKS, §4.1]. For an exceptional factor E of the s.o.d., we have a direct sum factor $K_0(E) = \mathbb{Z}$ of $K_0(Y)$. In this way, the maximal possible length is the rank of $K_0(Y)_{\mathbb{Q}}$. By [F2, Corollary 18.3.2], we have an isomorphism between $K_0(Y)_{\mathbb{Q}}$ and the Chow ring of Y over $\mathbb{Q}$, and so the maximum possible length of an exceptional collection is at least $\chi_{\text{top}}(Y)$ when $p_g(Y) = q(Y) = 0$. If the Bloch conjecture for $p_g = q = 0$ surfaces is true, then this maximum possible length is $\chi_{\text{top}}(Y)$. (We note that the Bloch conjecture is unknown only for surfaces of general type.) By a result of Vial, we will soon see that the topology of the surface Y restricts even more the maximal possible length.

Remark 6.8. If Y is rational, then we have the concrete isomorphism of groups [KKS, Lemma 4.2]

$$(\mathrm{Rank}, c_1, \chi)\colon K_0(Y) \to \mathbb{Z} \oplus \mathrm{Pic}(Y) \oplus \mathbb{Z},$$

where $\mathbb{Z} \oplus \mathrm{Pic}(Y) \oplus \mathbb{Z}$ is called the *Mukai lattice* of Y.

The following proposition will be the base for the general construction of Hacking exceptional collections from W-surfaces (see [H5, Lemma 2.5.2]).

Proposition 6.9. *Let Y be a nonsingular projective surface with $p_g(Y) = q(Y) = 0$. Let $\{\Gamma_1, \ldots, \Gamma_r\}$ be a chain of nonsingular rational curves on Y. (This is, $\Gamma_i \cdot \Gamma_{i+1} = 1$ and $\Gamma_i \cdot \Gamma_j = 0$ for any other $i \neq j$.) Then*

$$\mathcal{O}_Y(-\Gamma_r - \Gamma_{r-1} \ldots - \Gamma_1), \ldots, \mathcal{O}_Y(-\Gamma_2 - \Gamma_1), \mathcal{O}_Y(-\Gamma_1), \mathcal{O}_Y$$

is an exceptional collection.

Proof. Since $p_g(Y) = q(Y) = 0$, every line bundle on Y is exceptional. For $k \geq 1$ we have $\mathrm{Ext}^j(\mathcal{O}_Y(-\Gamma_i \ldots - \Gamma_1), \mathcal{O}_Y(-\Gamma_{i+k} \ldots - \Gamma_1)) = H^j(\mathcal{O}_Y(-\Gamma_{i+k} \ldots - \Gamma_{i+1})) = 0$ for every j as $p_g(Y) = q(Y) = 0$ and $\sum_{l=1}^{i} \Gamma_l$ is a connected curve of arithmetic genus 0. □

Remark 6.10. The previous e.c. is dual to $\mathcal{O}_Y, \mathcal{O}_Y(\Gamma_1), \ldots, \mathcal{O}_Y(\Gamma_r + \ldots + \Gamma_1)$, which is also e.c. The point of presenting it as in Proposition 6.9 is to put it in the exact form that will work for W-surfaces. See discussion in [TU, §4].

Example 6.11 (Many e.c. from chains)**.** Proposition 6.9 constructs maximal exceptional collections of line bundles in many surfaces. Take two lines in $\mathbb{P}^2$. Then $\mathcal{O}(-2), \mathcal{O}(-1), \mathcal{O}$ is maximal length (and so full because it is a del Pezzo). In $\mathbb{F}_m$ we can take two fibers of $\mathbb{F}_m \to \mathbb{P}^1$ and one section, and then we have a length 4 maximal collection. Similarly for any toric nonsingular surface. In an Enriques surface any nonsingular rational curve is a (-2)-curve. We have that the Picard number is 10, so we can have at most 9 curves in a chain, whose contraction is a $\mathbb{Q}$-homology projective plane with $K \equiv 0$ and one A_9 singularity. By Schütt [S4], there is a one-dimensional irreducible family of such "A_9 Enriques surfaces". One can see part of this family from W-surfaces with W a Coble surface, see details in [U2]. Similarly for elliptic surfaces with Kodaira dimension 1 and $p_g = q = 0$. One can construct W-surfaces that produce these elliptic fibrations as general fibers W_t together with a chain of 9 $\mathbb{P}^1$s. The case of general type is harder, but there are many examples of chains of $\mathbb{P}^1$s. If Y is of general type with $p_g = q = 0$, then we can have at most $9 - K_Y^2$ $\mathbb{P}^1$s in a chain.

Consider on $K_0(Y)$ the Euler pairing (bilinear form)

$$\chi(A, B) := \sum (-1)^i \,\mathrm{ext}^i(A, B),$$

where $A, B \in K_0(Y)$. We define the *maximum length of an exceptional collection* as the rank of $K_0^{\mathrm{num}}(Y) := K_0(Y)/\ker \chi$ (see for example [V2]). The Euler pairing allows us to talk about numerical exceptional collections, as a test to be an actual one. A vector bundle E is *numerically exceptional* if $\chi(E, E) = 1$. A collection of vector bundles $E_r, E_{r-1}, \ldots, E_0$ is a *numerical exceptional collection* (n.e.c.) if $\chi(E_i, E_i) = 1$ for all i, and $\chi(E_i, E_j) = 0$ for all $i < j$.

In [P3], Perling proves that a n.e.c. (of exceptional objects) of maximal length can be transformed into an n.e.c. of rank 1 objects. On the other hand, Vial [V2, Proposition 1] shows that this last part is equivalent to having a collection of divisors that mimics the construction above of an e.c. from a chain of rational curves. See also [V2, Theorem 6] for another equivalence. Using that, Vial proves the next classification theorem.

Definition 6.12. Let $n \geq 2$. A *Dolgachev surface* $D_{p_1,\ldots,p_n}$ is a minimal elliptic fibration over $\mathbb{P}^1$ with n multiple fibers of multiplicity p_i, and geometric genus 0.

Theorem 6.13 (Theorem 3 in [V2]). *Let Y be a nonsingular projective surface with $p_g = q = 0$. Let $\kappa(Y)$ be the Kodaira dimension of Y. If Y is not minimal, then Y admits a n.e.c. of maximal length. For Y minimal we have:*

- *If $\kappa(Y) = -\infty$, then Y admits a n.e.c. of maximal rank.*
- *If $\kappa(Y) = 0$, then Y is an Enriques surface and its maximal possible length is 10.*
- *If $\kappa(Y) = 1$, then Y is a Dolgachev surface, and it admits a n.e.c. of maximal length 12 if and only if Y is $D_{2,3}$, $D_{2,4}$, $D_{3,3}$ or $D_{2,2,2}$.*
- *If $\kappa(Y) = 2$, then Y admits a n.e.c. of maximal length.*

Cho and Lee constructed in [CL] an exceptional collection of line bundles of maximal length 12 with a phantom, i.e. it is not full, on a Dolgachev surface $D_{2,3}$. In [TU, Section 8] the authors construct exceptional collections of vector bundles of maximal possible length 10 on any $D_{p,q}$ with $\gcd(p,q) = 1$. They are all Hacking exceptional collections, and we explain them now.

Theorem 6.14 (Theorems 5.5 and 5.8 [TU]). *Let $W_t \rightsquigarrow W$ be a W-surface such that*

(1) $p_g(W) = q(W) = 0$.

(2) The surface W has exactly the Wahl singularities $P_0, \ldots, P_r$ (we also allow P_i to be smooth points), and a chain of nonsingular rational curves $\Gamma_1, \ldots, \Gamma_r$ that are toric boundary divisors Γ_i, Γ_{i+1} at P_{i+1}. (This is as in Figure 20 without Γ_0 and Γ_{r+1}.)

(3) There exists a Weil divisor $A \subset W$, which is Cartier outside of P_0 and generates the local class group $\mathrm{Cl}(P_0 \in W)$.

Then, after possibly shrinking $\mathbb{D}$, there exists an e.c. $E_r, \ldots, E_0$ of Hacking vector bundles on W_t with

$$\mathit{rank}(E_i) = n_i, \quad c_1(E_i) = -n_i(A + \Gamma_1 + \ldots + \Gamma_i) \in H_2(W_t),$$

where $P_i = \frac{1}{n_i^2}(1, n_i a_i - 1)$. (For P_i smooth we take $n_i = a_i = 1$.)

Definition 6.15. A *Hacking exceptional collection* (H.e.c.) on a nonsingular projective surface Y is the existence of a W-surface $W_t \rightsquigarrow W$ where $Y = W_t$ for some $t \neq 0$, and an e.c. as in Theorem 6.14 on Y.

Remark 6.16. By Riemann–Roch, we also have $c_2(E_i) = \frac{n_i - 1}{2n_i}(c_1(E_i)^2 + n_i + 1)$.

The condition (3) in Theorem 6.14 is satisfied in many cases by the next lemma.

Lemma 6.17 (Lemma 8.1 of [TU]). *Let Z be a surface with only c.q.s. $\{Q_0, \ldots, Q_s\}$ of type $\frac{1}{m_i}(1, q_i)$, and with $H^1(Z, \mathcal{O}_Z) = H^2(Z, \mathcal{O}_Z) = 0$. Let $Z^o := Z \backslash \{Q_0, \ldots, Q_s\}$. If $H_1(Z^o, \mathbb{Z}) = 0$, then there is a short exact sequence*

$$0 \to \mathrm{Pic}(Z) \to \mathrm{Cl}(Z) \to \bigoplus_{i=0}^{s} \mathrm{Cl}(Q_i \in Z) \to 0,$$

where $\mathrm{Cl}(Q_i \in Z) \simeq \mathbb{Z}/m_i\mathbb{Z}$ is the local class group of $Q_i \in Z$.

As shown in [TU, §8], one can use the Seifert-Van Kampen theorem to compute $\pi_1(W^o)$. If trivial, then Lemma 6.17 applies, and we have condition (3) in Theorem 6.14. See also [KKS], [KPS] for a direct relation with the vanishing of the Brauer group $\mathrm{Br}(W)$.

Remark 6.18. The existence of such an exceptional collection was stated in [H5, Theorem 2.5.1] for dual bundles (without a proof).

Remark 6.19. Let Y be a del Pezzo surface with a full exceptional collection $E'_r, \ldots, E'_0$ of vector bundles. In [H5, Theorem 2.5.3] it is stated that there exists a W-surface $W_t \rightsquigarrow W$ where W_t is a del Pezzo surface deformation equivalent to Y, and W is a toric surface, such that it induces a H.e.c. deformation equivalent to $E'_r, \ldots, E'_0$. By [KO], this implies that every exceptional bundle on a del Pezzo surface arises as a Hacking vector bundle [H5, Corollary 2.5.4].

Exercises.

(1) Reproduce the construction in Remark 6.19 via Theorem 6.14.
(2) Let $n \geq 2$ be an integer. One can construct Dolgachev surfaces $D_{n,n}$ (Definition 6.12; for $n = 2$ they are Enriques surfaces) by means of $\mathbb{Q}$-Gorenstein smoothings over two fibers in a rational elliptic surface with sections. See, for example, [U3, Section 4], the index of the corresponding Wahl singularities is n. Take, for example, the rational elliptic surface with 3 sections and singular fibers $I_9 + 3I_1$. Construct the T-singularity with $d = 10$ and index n from this configuration and a section, so that its $\mathbb{Q}$-Gorenstein smoothing is a $D_{n,n}$. There is a 1-dimensional family that produces a $D_{n,n}$ with a chain of 9 (-2)-curves. This defines a length 10 e.c. of line bundles. Write down the details.
(3) In [TU, §8] the authors construct a maximal length H.e.c. for all simply-connected surface $D_{p,q}$ (except for $p = 2, q = 3$, where the maximal possible length is 12). The starting point is the rational elliptic surface with 3 sections and singular fibers $I_9 + 3I_1$. Can it be done starting with rational elliptic surfaces with singular fiber configurations $2I_5 + 2I_2$ and $I_8 + 2I_1 + I_2$? Is it possible to extend the constructed H.e.c. of length 10 to a e.c. of length 12?
(4) For Enriques surfaces the condition (3) in Theorem 6.14 fails. Show it. On the other hand, there are length 10 exceptional collections of line bundles. Are there e.v.b. of rank two on Enriques surfaces? (Answer is yes; see for example [B4]. However, they cannot be Hacking vector bundles, since when K is trivial, $c_1 \cdot K = 0$ but for Hacking bundles $c_1(E) \cdot K = \pm a \bmod n$.)

6.3. **Exceptional collections and H.e.c.s.** Our main reference here is [B3]. Let Y be a nonsingular projective surface. An *exceptional object* in $D^b(Y)$ is a $A \in D^b(Y)$ such that $\mathrm{Hom}_{D^b(Y)}(A, A[i]) = 0$ for $i \neq 0$ and $\mathrm{Hom}_{D^b(Y)}(A, A) = \mathbb{C}$. (We need to consider them because operations on e.c. of vector bundles give e.c. but not necessarily of vector bundles.) An *exceptional collection* $A_r, \ldots, A_0$ is a collection of exceptional objects A_i such that $\mathrm{Hom}_{D^b(Y)}(A_i, A_j[k]) = 0$ for all $i < j$, and all k. This coincides with Definition 6.5 when the A_i are vector bundles on Y.

As we saw in Proposition 6.6, we have certain operations on exceptional collections. The following is another operation. From an exceptional collection $\langle A, B\rangle \subset D^b(Y)$ we can obtain two other exceptional collections $\langle B, R_B(A)\rangle$ (*right mutation* of A over B), and $\langle L_A(B), A\rangle$ (*left mutation* of B over A), so that $\langle A, B\rangle = \langle B, R_B(A)\rangle = \langle L_A(B), A\rangle$ (see for example [B3, §2]).

For a longer exceptional collection $\langle A_r, \ldots, A_0\rangle$, the action of left and right mutations induces an action of the braid group B_{r+1} (see Subsection 5.2) of $r + 1$ strands on $\langle A_r, \ldots, A_0\rangle$. Let us denote them by R_i (right mutation over the ith object) and by L_i

(left mutation over the ith object). In particular, a mutation of an e.c. is an e.c., and they generate the same category [B3, Lemma 2.2]. Hence, we also have

- $R_iL_i = 1$.
- $R_iR_{i+1}R_i = R_{i+1}R_iR_{i+1}$ and $L_iL_{i+1}L_i = L_{i+1}L_iL_{i+1}$.

Rational surfaces. Exceptional collections have been studied extensively on del Pezzo surfaces [DLP, GR, R3, G3, KO]. We start by highlighting two general results.

Let Y be a del Pezzo surface. We define the slope of a vector bundle E as $\mu(E) := -K_Y \cdot \frac{c_1(E)}{\text{rank}(E)}$.

Theorem 6.20 (2.4 Theorem [G3]). *Any e.v.b. E on a del Pezzo surface is Mumford-Takemoto stable, that is, for any coherent subsheaf $0 \neq F \subset E$ with $0 < rank(F) < rank(E)$ we have $\mu(F) < \mu(E)$.*

Theorem 6.21 (2.5 Corollary [G3]). *An exceptional vector bundle E on a del Pezzo surface Y is uniquely determined up to isomorphism by its slope $\frac{c_1(E)}{rank(E)} \in \text{Pic}(Y)_{\mathbb{Q}}$.*

We note that by [G3, 2.2.3 Corollary] $c_1^2(E)$ and $\text{rank}(E)$ of Theorem 6.21 are coprime.

Theorem 6.22 (2.11 Corollary [KO]). *Let E, F be an e.c. on a del Pezzo surface. Then $\text{Ext}^2(E,F) = 0$, and at most one $\text{Ext}^i(E,F) \neq 0$ for some $i = 0, 1$.*

Let E, F be vector bundles on a nonsingular projective surface Y. Then by the Riemann-Roch theorem, we have

$$\chi(E,F) = \chi(F,E) + \text{rank}(F)c_1(E)\cdot K_Y - \text{rank}(E)c_1(F)\cdot K_Y.$$

Corollary 6.23. *Let E, F be an e.c. on a del Pezzo surface Y. Then*

(i) $\mu(E) \leq \mu(F)$ if and only if $\hom(E,F) = rank(F)rank(E)(\mu(F) - \mu(E))$.
(ii) $\mu(E) \geq \mu(F)$ if and only if $\text{ext}^1(E,F) = rank(F)rank(E)(\mu(E) - \mu(F))$.

Corollary 6.24. *Let $\mathcal{O}_Y, F$ be an e.c. on a del Pezzo surface Y. Then*

(i) $0 \leq \mu(F)$ if and only if $h^0(Y,F) = rank(F)\mu(F)$.
(ii) $0 \geq \mu(F)$ if and only if $h^1(Y,F) = -rank(F)\mu(F)$.

The following is a complete picture for e.c. on $\mathbb{P}^2$.

Theorem 6.25 ([GR, R3]). *Up to tensoring and dualizing, every full exceptional collection of vector bundles E_2, E_1, E_0 on $\mathbb{P}^2$ can be obtained by mutating finitely many times the e.c. $\mathcal{O}(-2), \mathcal{O}(-1), \mathcal{O}$. The ranks of E_2, E_1, E_0 form a Markov triple, and a mutation of E_2, E_1, E_0 is a mutation on the corresponding Markov triple. (Thus for any Markov triple we have an e.c. with those ranks.)*

Let us show that any full e.c. is a H.e.c. For that, let $(1 < a < b < c)$ be a Markov triple. (The other cases, i.e. where $a = 1$, are simpler to describe.) Let us consider a W-surface $\mathbb{P}^2 \rightsquigarrow W$ with $W = \mathbb{P}(a^2, b^2, c^2)$. As In Subsection 4.1, we have

$$0 \to \text{Pic}(\mathbb{P}^2) \to \text{Cl}(W) \to \mathbb{Z}/a \oplus \mathbb{Z}/b \oplus \mathbb{Z}/c \to 0.$$

We recall that each $\mathbb{Z}/n$ corresponds to the first homology group of the Milnor fiber of a $\mathbb{Q}$-Gorenstein smoothing of the Wahl singularity $\frac{1}{n^2}(1, nq-1)$. The weighted projective plane $\mathbb{P}(a^2, b^2, c^2)$ has Wahl singularities $\frac{1}{a^2}(b^2, c^2)$, $\frac{1}{b^2}(c^2, a^2)$, and $\frac{1}{c^2}(b^2, a^2)$.

Definition 6.26. Given a Markov triple $(a < b < c)$ and $x \in \{a, b, c\}$, we define integers $0 < r_x, w_x < x$ as follows:

- $r_a \equiv b^{-1}c \pmod a$, $r_b \equiv c^{-1}a \pmod b$, and $r_c \equiv a^{-1}b \pmod c$.
- $w_a \equiv 3b^{-1}c \pmod a$, $w_b \equiv 3c^{-1}a \pmod b$, and $w_c \equiv 3a^{-1}b \pmod c$.

In our case, if $x \in \{a, b, c\}$ $(x \neq 1)$, then the corresponding Wahl singularity is $\frac{1}{x^2}(1, xw_x - 1)$. The following are basic properties (see for example [A1], [R3], [P4, Cor.5.4]).

Proposition 6.27. *Let $x > 1$ be part of a Markov triple $(a < b < c)$, then $x + w_x = 3r_x$, $r_x^2 \equiv -1 (\mathrm{mod}\ x)$, $r_c a - r_a c = b$, $cr_b - br_c = a$, and $ar_b - br_a = 3ab - c$.*

As in [M1, Theorem 18], the minimal resolution X of W can be thought of in a particular way. The surface X is a sequence of blow-ups from a Hirzebruch surface $\mathbb{F}_m \to \mathbb{P}^1$, so that the minimal resolution of $\frac{1}{a^2}(1, aw_a - 1)$ is contained in one fiber, the one of $\frac{1}{b^2}(1, bw_b - 1)$ in another fiber, and the one of $\frac{1}{c^2}(1, cw_c - 1)$ contains curves in both fibers together with the strict transform of the negative curve in $\mathbb{F}_m$. In this way, there are unique (-1)-curves Γ_1 and Γ_2, one for each of these two special fibers respectively, which are not part of the exceptional divisor of the minimal resolution of W.

Lemma 6.28. *We can write $\mathrm{Cl}(W) = \langle \Gamma_1, \Gamma_2 \rangle / (a^2\Gamma_1 - b^2\Gamma_2)$, which is generated by the class*

$$\zeta := a'\Gamma_1 + b'\Gamma_2$$

where a', b' is a solution of $a'b^2 + b'a^2 = 1$. In this way $\Gamma_1 = b^2\zeta$ and $\Gamma_2 = a^2\zeta$. We have $\zeta^2 = \frac{1}{a^2b^2c^2}$ and $-K_{W_0} = 3abc\zeta$.

Proof. One can show that the multiplicities of Γ_1 and Γ_2 in the corresponding fibers are a^2 and b^2 respectively. Therefore, we have that the class group of W_0 is isomorphic to $\langle \Gamma_1, \Gamma_2 \rangle$ quotient by $(a^2\Gamma_1 - b^2\Gamma_2)$, which is the relation given by the fibers. Let us find $a'\Gamma_1 + b'\Gamma_2$ so that

$$\langle a'\Gamma_1 + b'\Gamma_2, a^2\Gamma_1 - b^2\Gamma_2 \rangle = \langle \Gamma_1, \Gamma_2 \rangle.$$

For this we just solve the equation $a'b^2 + b'a^2 = 1$. The rest is a trivial check. □

Lemma 6.29. *Consider Γ_1 as part of a toric boundary at $\frac{1}{a^2}(1, aw_a - 1)$. Then we can take*

$$A := bcd\,\zeta$$

where $d = r_a + as$, and $s \in \mathbb{Z}$, as its complementary toric boundary, which is Cartier at the other two singularities.

Proof. If we want $\alpha\zeta$ to be trivial at $\mathbb{Z}/b$ and $\mathbb{Z}/c$, then b and c must divide α, as a, b, c are coprime. If we want $bcd\zeta$ to be complementary toric boundary to A at $\frac{1}{a^2}(b^2, c^2)$, then we need $bcda' \equiv -1$ modulo a. Using the equations

$$a'b^2 + b'a^2 = 1 \quad \text{and} \quad a^2 + b^2 + c^2 = 3abc$$

we obtain that $d = a'cb + as$, where $s \in \mathbb{Z}$. One shows that $r_a \equiv a'cb$ modulo a. □

We have now all the hypothesis of Theorem 6.14, this is, we have the chain Γ_1, Γ_2 of smooth rational curves passing as toric boundaries through the singularities, and the existence of A. Therefore, we have the exceptional collection of vector bundles

$$E_b, E_c, E_a$$

of ranks b, c, a respectively, such that

$$c_1(E_a^\vee) = aA, \quad c_1(E_c^\vee) = c(A+\Gamma_1), \quad c_1(E_b^\vee) = b(A+\Gamma_1+\Gamma_2),$$

using the identification of Pic$(\mathbb{P}^2)$ as the kernel of $\mathrm{Cl}(W) \to \mathbb{Z}/a \oplus \mathbb{Z}/b \oplus \mathbb{Z}/c$. By Riemann-Roch $c_2(E_x^\vee) = \frac{x-1}{2x}(c_1(E_x^\vee)^2 + x + 1)$.

Using Proposition 6.27 we have $c_1(E_a^\vee) \cdot H = r_a + as$, $c_1(E_c^\vee) \cdot H = r_c + cs$, and $c_1(E_b^\vee) \cdot H = r_b + bs$.

A trivial observation is that for any $x \in \{a, b, c\}$ we have $r_x^2 \equiv -1$ modulo x. Therefore r_x/x or $(x - r_x)/x$ is in the interval $]0, 1/2]$.

Theorem 6.30. *Every full e.c. in $\mathbb{P}^2$ is a H.e.c.*

Proof. By [R3] we have that (r_a, r_b, r_c) (together with the ranks) determines a full exceptional collection on $\mathbb{P}^2$. □

In fact, one can prove the following (see [H5, Theorem 2.5.3]; see also [UZ2, §8]).

Theorem 6.31. *Every full e.c. (of vector bundles) on a del Pezzo surface is a H.e.c.*

There are various related results in [UZ2].

Remark 6.32. (Mutations) By [GR, R3], we saw that any mutation of Markov numbers corresponds to a mutation of the corresponding full e.c. For each of them we have W-surfaces $\mathbb{P}^2 \rightsquigarrow \mathbb{P}(a^2, b^2, c^2)$ and $\mathbb{P}^2 \rightsquigarrow \mathbb{P}(a^2, b^2, c'^2)$ where $c' = 3ab - c$. It turns out that these degenerations are connected by a deformation over $\mathbb{P}^1$, where each of them corresponds one points in $\mathbb{P}^1$, and over the complement we have partial smoothings of the singularities corresponding to c and c' (see [H4, Example 6.3]).

Enriques surfaces. There are constructions of exceptional vector bundles on Enriques surfaces (e.g. [Z2]), but they cannot be Hacking vector bundles. The reason is that for a Hacking E from a W-surface $W_t \rightsquigarrow W$ we must have $K_{W_t} \cdot c_1(E) \equiv \pm a$ modulo n, where $\frac{1}{n^2}(1, na-1)$ is the Wahl singularity. Therefore, K_{W_t} cannot be numerically trivial, and so W_t cannot be an Enriques surface. In general, if n is the rank of E, then a necessary condition is $K_{W_t} \cdot c_1(E)$ to be coprime to n.

Dolgachev surfaces. This means the case of Kodaira dimension 1. Here we can construct many H.e.c. through configurations of rational curves from elliptic fibrations. First see [U3, Section 4] to construct Dolgachev surfaces from rational elliptic surfaces. Then see [TU, Section 8] to see a detailed example for any simply-connected Dolgachev surface. By Theorem 6.13, we have very few cases with maximal length.

Surfaces with big canonical class. Surfaces of general type with $p_g = q = 0$ are harder to construct, but there are plenty of examples via W-surfaces [LP1, PPS1, PPS2]. By Theorem 6.13, there are always numerically maximal e.c. There are some examples in the literature, but very little is known. Let W be a surface with no obstructions to deform, only Wahl singularities, and K_W big and nef. Then $K_W^2 = 1, 2, 3, 4$. Hence, the maximal possible lengths are $11, 10, 9, 8$ respectively, but the maximum number of Wahl singularities is $8, 6, 4, 2$ respectively. Therefore, if the recipe in Theorem 6.14 works for a maximal amount of singularities, then we would obtain that the Mukai lattices of the complement of the H.e.c. have ranks $3, 4, 5, 6$ respectively.

We end this last section stating two theorems from [TU] in relation to a semi orthogonal decomposition of the 3-fold in a $\mathbb{Q}$-Gorenstein smoothing and its restrictions to the fibers, and to certain cohomological properties of the H.e.c.s of an M-resolution and an its N-resolution of a given c.q.s. See [TU] for notations.

Theorem 6.33 (Theorem 1.12 [TU])**.** *Let $Y \rightsquigarrow W$ be a $\mathbb{Q}$-Gorenstein smoothing of a surface W with only Wahl singularities satisfying Assumption 5.12 (1), (2), (3). After possibly shrinking the base B, $D^b(\mathcal{W})$ admits a B-linear*[10] *s.o.d. $\langle \mathcal{A}_r^{\mathcal{W}}, \dots, \mathcal{A}_0^{\mathcal{W}}, \mathcal{B}^{\mathcal{W}} \rangle$ compatible with respect to restrictions to W and Y*

$$\langle \mathcal{A}_r^W, \dots, \mathcal{A}_0^W, \mathcal{B}^W \rangle \xleftarrow{Li_W^*} \langle \mathcal{A}_r^{\mathcal{W}}, \dots, \mathcal{A}_0^{\mathcal{W}}, \mathcal{B}^{\mathcal{W}} \rangle \xrightarrow{Li_Y^*} \langle \mathcal{A}_r^Y, \dots, \mathcal{A}_0^Y, \mathcal{B}^Y \rangle. \tag{6.1}$$

Each $\mathcal{A}_i^Y$ is generated by the Hacking bundle E_i and and each $\mathcal{A}_i^W \simeq D^b(R_i\text{-mod})$, where R_i is the Kalck-Karmazyn algebra associated to $P_i \in W$. Furthermore, $\mathcal{B}^{\mathcal{W}} \subset D^{\mathrm{perf}}(\mathcal{W})$.

Theorem 6.34 (Theorem 1.13 [TU])**.** *Let W^+ be an M-resolution of $P \in \overline{W}$ satisfying Assumption 5.12. Fix a $\mathbb{Q}$-Gorenstein smoothing $Y \rightsquigarrow W^+$ which is sufficiently general in its irreducible component of the versal deformation space of $\overline{W}$. This component also contains a $\mathbb{Q}$-Gorenstein smoothing $Y \rightsquigarrow W^-$, where W^- is the N-resolution associated to W^+.*

(1) Let $\bar{E}_r, \dots, \bar{E}_0$ be a Hacking exceptional collection on Y associated with the N-resolution W^-. This collection is strong: $\mathrm{Ext}^k(\bar{E}_i, \bar{E}_j) = 0$ for $k > 0$ and $i > j$.
(2) In contrast, let $E_r, \dots, E_0$ be a Hacking exceptional collection on Y associated with the M-resolution W^+. Then we have $\mathrm{Ext}^k(E_i, E_j) = 0$ for $k \neq 1$ and $i > j$.
(3) For $i = 1, \dots, r$, we have $\mathrm{Hom}(\bar{E}_{r+1-i}, \bar{E}_{r-i}) \simeq \mathrm{Ext}^1(E_i, E_{i-1})^\vee \simeq \mathbb{C}^{\delta_i}$.
(4) The Kawamata bundle $\bar{F}$ on $\overline{W}$ deforms to a vector bundle $F \simeq \bigoplus_{i=0}^{r} \bar{E}_i^{n_{r-i}}$ on Y. Since F has rank Δ, we note that

$$\Delta = n_0\bar{n}_r + n_1\bar{n}_{r-1} + \ldots + n_r\bar{n}_0.$$

(5) The Kalck–Karmazyn algebra $\bar{R} = End(\bar{F})$ deforms to the algebra $End(F)$, which is hereditary and Morita-equivalent to the path algebra $\hat{R} = End(\bar{E}_r \oplus \ldots \oplus \bar{E}_0)$. The E_i correspond to simple $\hat{R}$-modules, and the $\bar{E}_i$ correspond to the indecomposable projective $\hat{R}$-modules.

An interesting fact in relation to quiver algebras is the following. Theorem 6.34 gives a large amount of admissible embeddings of derived categories $D^b(\hat{R}\text{-mod})$ of acyclic quivers without relations into derived categories of smooth projective surfaces Y (which can be chosen to be rational). (In general, the number of irreducible components of $\mathrm{Def}_{P \in \overline{W}}$ is the sth Catalan number, where s is the length of the corresponding dual continued fraction.) Although Orlov proved [O2] that the embedding always exists if $\dim Y$ is sufficiently large, there are strong restrictions in the case of surfaces. In fact, very few examples were known before [TU]. In particular, Belmans and Raedschelders [BR, Sect.4] asked whether there are bounds on the lengths of paths of realizable quivers, and which acyclic quivers $Q_{a,b,c}$ with 3 vertices, where a, b, c are the number of arrows between them, are realizable. Are they all realizable? In [TU, Prop. 6.11] It is shown that lengths of paths are unbounded. For the other, see the exercises.

Exercises.

(1) In relation to the second question of Belmans and Raedschelders [BR, Sect.4], in [TU] the following theorem is proved.

[10] I.e. preserved by tensoring with a pullback of any object $T \in D^{\mathrm{perf}}(B)$.

Theorem 6.35. *The quiver $Q_{a,b,c}$ is realizable by the algebra $\hat{R}$ if and only if there exists an extremal P-resolution with Wahl singularities of indices a and b and with $\delta = c$.*

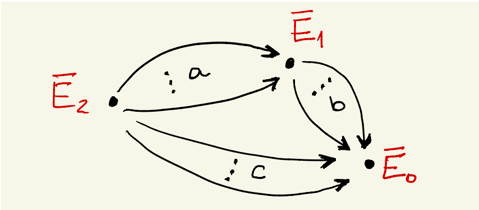


FIGURE 25. The quiver $Q_{a,b,c}$ for an N-resolution with two curves

The list of realizable triples (a, b, c) from this theorem with entries smaller than or equal to 8 is:

(1, 2, 3) (1, 2, 5) (1, 2, 7) (1, 3, 4) (1, 3, 5) (1, 3, 7) (1, 3, 8) (1, 4, 5)
(1, 4, 7) (1, 5, 6) (1, 5, 7) (1, 5, 8) (1, 6, 7) (1, 7, 8) (2, 2, 4) (2, 2, 8)
(2, 3, 5) (2, 3, 7) (2, 4, 6) (2, 5, 7) (2, 6, 8) (3, 3, 3) (3, 3, 6) (3, 4, 5)
(3, 4, 7) (3, 5, 7) (3, 5, 8) (3, 7, 8) (4, 4, 8) (4, 5, 7) (5, 5, 5) (5, 6, 7)
(5, 7, 8) (7, 7, 7).

Show that all extremal P-resolutions as in Theorem 6.35 are listed as:
(0) If $a = b = 1$, then $c = \lambda - 1$,
(1) If $a = 1$ and $b > 1$, then $c = \lambda b - b - \epsilon_b$; If $b = 1$ and $a > 1$, then $c = \lambda a - a - \epsilon_a$,
(2) If $a, b > 1$, then $c = (\lambda - 1)ab - \epsilon_a b - \epsilon_b a$. In particular, gcd$(a, b)$ always divides c.

(2) $\star\star$ In the previous question/list we have all triples that are realizable via H.e.c. of length 3 over a c.q.s. What happens with the resting triples (a, b, c)? For example, what can you say about $(2, 2, 2(2t + 1))$?

(3) $\star\star$ In the case of $p_g = q = 0$ surfaces of general type and when the Kodaira dimension is 1: Which Mukai lattices do appear for the complement of maximal possible length H.e.c.? In the case of general type, one needs to find many examples with the maximum possible number of Wahl singularities, and forming a chain.

(4) $\star\star$ The associated non-commutative deformations of the Kalck-Karmazin algebra $\bar{R}$ in Theorem 6.34 are not explicit. Find them. There is now a much better way to write down $\bar{R}$ [LT], which may be useful for this question. Particularly, check the beautiful [LT, Conjecture 1.9].

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