SGA 2022-1 (Andrei Neguț): "The Beauville-Voisin conj for $\mathrm{Hilb}^n(K3)$" (1)

(joint work with Davesh MAULIK).

Hyperkähler manifolds $\Leftrightarrow$ Complex manifold X endowed with a non-degenerate holom. symplectic 2-form

$$\omega : T_X \otimes T_X \to \mathbb{C} \quad , \quad \omega(a,b) = -\omega(b,a)$$

Induces an isom $T_X \xrightarrow{\sim} T_X^*$, $v \mapsto \omega(v, \cdot)$

This implies $\dim_{\mathbb{C}} X$ = even

Def: A HSV (holom. symplectic variety) is an smooth algebraic variety $X/\mathbb{C}$ endowed with a non-degenerate algebraic symplectic 2-form.

Today: We will assume X is projective.

- No examples of HSV in dim = 1.
- In dim 2, only two examples: $\underbrace{\text{complex tori (abelian variety)}}_{\text{NOT simply connected}}$ & $\underbrace{\text{K3 surfaces}}_{\text{simply connected}}$

From now on, we also assume $\pi_1(X) = \{1\}$.

Beauville constructed the first known family of HSV, namely:

$\mathrm{Hilb}^n(S)$ = Hilbert scheme of n points on a K3 surface S.

(Mukai constructed another family, namely moduli spaces of stables sheaves on S.)

The only other examples of HSV known are two examples due to O'Grady in dim 6 and dim 10.

$$\mathrm{Hilb}^n(S) = \{ \text{ideals } I \subseteq \mathcal{O}_S, \text{ length } \mathcal{O}_S/I = n \}$$

$I := \ker \varphi$

$$= \mathrm{Quot}_{\mathcal{O}_S \overset{\varphi}{\twoheadrightarrow} \mathcal{E}} \quad \text{Hilbert pol. of } \mathcal{E} \text{ is } n$$

Def: Let X proj var, $\mathcal{V}$ a coherent sheaf on X, $P(t)$ a polynomial.

$$\mathrm{Quot}_{\mathcal{V} \twoheadrightarrow \mathcal{E}, P(t)} = \left\{ \begin{array}{l} \text{quotients } \mathcal{V} \twoheadrightarrow \mathcal{E}, \text{ where} \\ \mathcal{E} \text{ is coherent sheaf st} \\ \dim H^0(X, \mathcal{E} \otimes \mathcal{O}(t)) = P(t) \text{ for } t \gg 0 \end{array} \right\}$$

$\mathrm{Hom}(\underset{\substack{\uparrow \\ \text{arbitrary scheme}}}{T}, \mathrm{Quot}_{\mathcal{V} \twoheadrightarrow \mathcal{E}, P(t)}) = \left\{ \begin{array}{l} \text{coherent sheaves } \mathcal{E} \to T \times X \text{ flat over } T \\ \text{whose fibers over any } p \in T \text{ have Hilbert poly} \\ P(t) \text{, plus a map } \pi^* \mathcal{V} \twoheadrightarrow \mathcal{E} \text{, where} \\ \pi : T \times X \longrightarrow X \end{array} \right\}$

Tangent spaces of Quot are

$$\mathrm{Hom}(\mathrm{Spec}\, \mathbb{C}[\varepsilon]/\varepsilon^2, \mathrm{Quot}_{\mathcal{V} \overset{\varphi}{\twoheadrightarrow} \mathcal{E}, P(t)}) \underset{\substack{\uparrow \\ \boxed{\text{Exercise}}}}{=} \mathrm{Hom}_X(\ker \varphi, \mathcal{E})$$

For the Hilbert scheme: $\mathrm{Hilb}^n(S)$, the tangent space to a closed point $I \in \mathrm{Hilb}^n(S)$ is $\mathrm{Hom}_S(I, \mathcal{O}_S/I)$.

Construction of the symplectic form:

1° $\mathrm{Hom}_S(I, \mathcal{O}_S/I) \cong \mathrm{Ext}^1_S(I, I)$ coming from $0 \to I \to \mathcal{O}_S \to \mathcal{O}_S/I \to 0$

2° $\dim \mathrm{Hom}_S(I, \mathcal{O}_S/I) = 2n \Rightarrow \mathrm{Hilb}^n(S)$ smooth of dim $2n$.

3° $\mathrm{Ext}^1(I, I) \otimes \mathrm{Ext}^1(I, I) \xrightarrow{\omega} \mathrm{Ext}^2(I, I)$

$\quad\quad \cong$ Serre duality

$\quad\quad \mathrm{Hom}(I, I \otimes K_S)$

$\quad\quad\quad \overset{\shortparallel}{\mathcal{O}_S}$ because S is K3

$\quad\quad \cong$

$\quad\quad \mathbb{C}$ (since $\mathrm{Hom}_S(I, I) \simeq \mathbb{C}$ & $\mathrm{Hom}_S(\mathcal{O}_S, \mathcal{O}_S) \simeq H^0(S, \mathcal{O}_S) \simeq \mathbb{C}$)

4° ω is a non-degenerate symplectic 2-form.

Let X be proj smooth variety$/\mathbb{C}$:

$A^*(X)$ ↗ alg cycles on X with $\mathbb{Q}$ coefficients $\sum r_i [Z_i]$, $r_i \in \mathbb{Q}$, Z_i alg sub. of $X/\underset{\text{rat}}{\sim}$ $\quad\quad [Z]$

$\eta \downarrow \quad\quad\quad\quad \downarrow$

$H^{2*}(X, \mathbb{Q}) \quad\quad$ Poincaré dual class of Z

↑ singular cohom. of X

$* = \mathrm{codim}\, Z \quad$ Eg. $A^0(X) = \mathbb{Q}$, $A^1(X) = \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \mathbb{Q}^{\rho(X)}$

Hodge conjecture: Any class in $H^{2k}(X, \mathbb{Q}) \cap H^{k,k}$ is in the image of η.

When S is K3

$$\begin{array}{ccc} \mathbb{Q} = A^0(S) & \xrightarrow[\sim]{\eta} & H^0(S,\mathbb{Q}) \simeq \mathbb{Q} \\ \oplus & & \oplus \\ \mathbb{Q}^{22-b} \simeq A^1(S) & \xhookrightarrow{\eta} & H^2(S,\mathbb{Q}) \simeq \mathbb{Q}^{22} \\ \oplus & & \oplus \\ \boxed{A^2(S)} & \overset{\eta}{\twoheadrightarrow} & H^4(S,\mathbb{Q}) \simeq \mathbb{Q} \end{array}$$

$b \in \{2,\dots,21\}$

very very infinite (Mumford !)

Beauville - Voisin : $\forall$ point p on any $\mathbb{P}^1 \hookrightarrow S$, $[p] \in A^2(S)$ does <u>not</u> depend on p.

Let's call this class $\theta \in A^2(S)$.

Recall that $A^*(X)$ is a ring, with $[Z]\cdot[Z'] \text{ "=" } [Z \cap Z']$.

$\Rightarrow \forall\, \ell, \ell' \in A^1(S)$, then $\ell \cdot \ell' = \underset{\text{number}}{\text{rational}} \cdot \theta$

Let $R(S) \subseteq A^*(S)$ be the subring generated by divisors (ie, $A^1(S)$)

$\Rightarrow R(S) = \mathbb{Q} \oplus A^1(S) \oplus \mathbb{Q}\theta \hookrightarrow H^{2*}(S)$

By the way: $c_1(S) = 0$, $c_2(S) = 24\,\theta$

$\Rightarrow R(S)$ is also the subring generated by divisors and Chern classes of T_S.
(resp. Beauville - Voisin)

<u>Beauville's conjecture</u>: $\forall$ HSV X, let $R(X) \subseteq A^*(X)$ be the subring generated by $A^1(X)$, then $\eta|_{R(X)}$ is injective, ie,
(resp. & $c_k(S)$ $\forall k$)

$$R(X) \longrightarrow H^{2*}(X,\mathbb{Q})$$

<u>Thm</u>: S K3 surface. Then, $X = \mathrm{Hilb}^n(S)$ verifies the Beauville's conj.

$$\underbrace{R(\mathrm{Hilb}^n(S))}_{} \hookrightarrow H^{2*}(\mathrm{Hilb}^n(S))$$

subring generated by tautological classes
w/ divisors ~~X~~ $c_k(S)$

$$\begin{array}{ccccc} & & \mathcal{I} & & \text{univ. ideal sheaf} \\ & & \downarrow & & \\ & & \mathrm{Hilb}^n(S)\times S & & \\ & \swarrow^{\pi_1} & & \searrow^{\pi_2} & \\ \mathrm{Hilb}^n(S) & & & & S \end{array}$$

$\rightsquigarrow \{\pi_{1*}(c_k(\mathcal{I})\cdot \pi_2^*\alpha)\}$
$\forall k \in \mathbb{N}$
$\forall \alpha \in R(S)$