D. Gajardo : Teorema de THOE 23/08/18 II (1)

Teo. de Thoe : Sea α algebraico grado d $\forall \varepsilon>0$, $\exists C=C(\alpha,\varepsilon)$ tal que $\forall \frac{p}{q}\in\mathbb{Q}$ $(p,q)=1$
$\Rightarrow \left|\alpha-\frac{p}{q}\right| \geq \frac{C(\alpha,\varepsilon)}{q^{\ell}}$, $\ell=\frac{d}{2}+1+\varepsilon$

Preliminares.

Lema 1 (Siegel): Dados $(a_{ij})\in\mathbb{Z}$, $1\leq i\leq q$, $1\leq j\leq p$ $(a_{ij})\leq A$ $\forall i,j$. Si $q>p$ el sistema $\sum_{i=1}^{q} a_{ij}x_i=0$

tiene una solución no trivial $(x_1,\dots,x_q)\in\mathbb{Z}^q$ $|x_i|\leq(2qA)^{\frac{p}{q-p}}$

obs|.- Si $q>(1+\delta)p \Rightarrow \frac{1}{q-p}<\frac{1}{\delta} \Rightarrow |x_i|\leq(2qA)^{\frac{1}{\delta}}$

Dem: $U=\{(x_1,\dots,x_q)\in\mathbb{Z}^q : 0\leq x_i\leq \underbrace{\lfloor(2qA)^{\frac{p}{q-p}}\rfloor}_{X}\}$

$V=\{(y_1,\dots,y_p)\in\mathbb{Z}^p ; y_j=\sum a_{ij}x_i , (x_1,\dots,x_q)\in U\}$

$|y_j|\leq\sum|a_{ij}||x_i|<AXq \Rightarrow |V|\leq(2XA+1)^p$ $|U|=(X+1)^q$

$\therefore |U|=(X+1)^q=(X+1)^q(X+1)^{p-q}>(X+1)^p(2qA)^p\geq(2qAX+1)^p=|V|$

$\therefore \exists (x_1,\dots,x_q),(x_1',\dots,x_q')\in U$ tal que $\underbrace{\sum a_{ij}x_i=\sum a_{ij}x_i'}$
$\sum a_{ij}(x_i-x_i')=0$ con sol. ~~sin~~ no trivial

Lema 2 : Sea α algebraico entero (pol. minimal $P_\alpha(x)=x^d+a_{d-1}x^{d-1}+\dots+a_0\in\mathbb{Z}[x]$)
$H=\text{máx}=\{1,|a_{d-1}|,\dots,|a_0|\}$, $\alpha^s=\sum_{t=0}^{d-1}b_{t,s}\alpha^t$ $b_{t,s}\in\mathbb{Z}$.

$\Rightarrow |b_{t,s}|\leq(2H)^s$

Dem: Inducción usar $\alpha^{s+1}=\alpha\cdot\alpha^s$.

obs|.- Si α alg $\exists N\in\mathbb{N}$ tq $N\alpha$ es alg entero.

Asumir α alg. entero.

Dem (THOE): $\alpha \in \mathbb{R}$, $\deg(\alpha) = d \geq 2$

$\boxed{\text{Paso 1}}$: Construir pol. auxiliar [Notación $R(x,y)$ pol, $R_m(x,y) = \frac{1}{m!}\frac{\partial^m R(x,y)}{\partial x^m}$]

Lema 3: Sean $\delta > 0$, $k, \ell \in \mathbb{N}$, $2(\ell+1) > (1+\delta)kd$

$\exists$ $R(x,y) \in \mathbb{Z}[x,y]$ no nulo $R(x,y) = \sum_{i=0}^{\ell}\sum_{j=0}^{1} a_{ij} x^i y^j$

tq $R_m(\alpha,\alpha) = 0$ $\forall 0 \leq m < k$, y $|a_{ij}| \leq C_0^{\ell}$, $C_0 = C_0(\alpha, \delta)$.

Dem: $R_m(x,y) = \sum_{i=0}^{\ell}\sum_{j=0}^{1} a_{ij}\binom{i}{m} x^{i-m} y^j$

$$0 = R_m(\alpha,\alpha) = \sum_{i=0}^{\ell}\sum_{j=0}^{1} a_{ij}\binom{i}{m}\alpha^{i+j-m} = \sum\sum a_{ij}\binom{i}{m}\sum_{t=0}^{d-1} b_{t,i+j-1}\alpha^t$$

$$\Rightarrow 0 = \sum_{t=0}^{d-1}\alpha^t \underbrace{\left(\sum_{i=0}^{\ell}\sum_{j=0}^{1} a_{ij}\binom{i}{m} b_{t,i+j-1}\right)}$$

$\Leftrightarrow$ Coeficientes son cero para cada t. $0 \leq t \leq d-1$, $0 \leq m \leq k-1$

Como $2(\ell+1) > (1+\delta)kd$ $\Rightarrow$ Lema 1 $\left|\binom{i}{m} b_{t,i+j-1}\right| \leq 2^i (2H)^{i+j-m}$
$\leq 2^{\ell}(2H)^{\ell+1} \leq (4H)^{2\ell}$.

Por Lema 1, los a_{ij} existen, son enteros no todos cero.
y $|a_{ij}| \leq (2\cdot 2(\ell+1)(4H)^{2\ell})^{1/\delta} \leq M^{\ell/\delta} = C_0^{\ell}$ ∎

$\boxed{\text{Paso 2}}$: Acotar $R_m\left(\frac{p_1}{q_1}, \frac{p_2}{q_2}\right)$ por arriba.

Lema 4: Sean δ, k, ℓ como en Lema 3 y sea $R(x,y)$ el pol. de Lema 3.
Sean $\frac{p_1}{q_1}, \frac{p_2}{q_2} \in \mathbb{Q}$ $(p_i, q_i) = 1$ $q_i > 0$ $\left|\alpha - \frac{p_i}{q_i}\right| < 1$ $i = 1,2$

$$\left|R_m\left(\frac{p_1}{q_1}, \frac{p_2}{q_2}\right)\right| \leq \left(\left|\alpha - \frac{p_1}{q_1}\right|^{k-m} + \left|\alpha - \frac{p_2}{q_2}\right|\right) C_1^{\ell} \quad C_1 = C_1(\alpha, \delta).$$

Dem: $R(x,y) = P(x) - y\,q(x)$

$R_m(x,y) = P_m(x) - y\,q_m(x) \qquad P_m(x) = \frac{1}{m!} P^{(m)}(x)$

vemos que $\left|P_m(x)\right| = \left|\sum a_{i,0}\binom{i}{m} x^{i-m}\right| \leq \sum_{i=0}^{\ell} c_0^{\ell} 2^{\ell}(1+|x|)^{i-m}$

$\therefore \quad |P_m(x)| \leq c_0^{\ell} 2^{\ell} (1+|x|)^{\ell}(\ell+1) \leq c_2^{\ell}(1+|x|)^{\ell}$

$|q_m(x)| \leq c_2^{\ell}(1+|x|)^{\ell}$

$\therefore \quad |R_m(x,y)| \leq |P_m(x)| + |y||q_m(x)| \leq c_2^{\ell}(1+|x|)^{\ell}(1+|y|)$.

$\left|R_m\left(\frac{p_1}{q_1}, \frac{p_2}{q_2}\right)\right| \leq \left|p_m\left(\frac{p_1}{q_1}\right) - \alpha\, q_m\left(\frac{p_1}{q_1}\right)\right| + \left|\left(\alpha - \frac{p_2}{q_2}\right) q_m\left(\frac{p_1}{q_1}\right)\right|$

$\left|\left(\alpha - \frac{p_2}{q_2}\right) q_m\left(\frac{p_1}{q_1}\right)\right| \leq \left|\alpha - \frac{p_2}{q_2}\right| c_2^{\ell}\left(1+\left|\frac{p_1}{q_1}\right|\right)^{\ell} \leq c_3^{\ell}\left|\alpha - \frac{p_2}{q_2}\right|$

$\left|p_m\left(\frac{p_1}{q_1}\right) - \alpha\, q_m\left(\frac{p_1}{q_1}\right)\right| = \left|R_m\left(\frac{p_1}{q_1}, \alpha\right)\right|$ pero

$$R_m(x,\alpha) = \sum_{t=0}^{\infty} \frac{1}{t!} R_m^{(t)}(x,\alpha)\Big|_{x=\alpha} (x-\alpha)^t = \sum_{t=0}^{\infty} \binom{m+t}{t} R_{m+t}(\alpha,\alpha)(x-\alpha)^t$$

$$\therefore \quad R_m(x,\alpha) = \sum_{t=k-m}^{\ell-m} \binom{m+t}{t} R_{m+t}(\alpha,\alpha)(x-\alpha)^t$$

$$\therefore \quad \left|R_m\left(\frac{p_1}{q_1}, \alpha\right)\right| \leq \sum_{t=k-m}^{\ell-m} 2^{m+t} c_2^{\ell}(1+|\alpha|)^{2\ell}\left|\frac{p_1}{q_1} - \alpha\right|^t \qquad \left|\frac{p_1}{q_1} - \alpha\right| < 1$$

$$\leq 2^{\ell} c_2^{\ell}(1+|\alpha|)^{2\ell}\left|\frac{p_1}{q_1} - \alpha\right|^{k-m}(\ell+1)$$

$$\leq c_4^{\ell}\left|\alpha - \frac{p_1}{q_1}\right|^{k-m}$$

$$\therefore \quad \left|R_m\left(\frac{p_1}{q_1}, \frac{p_2}{q_2}\right)\right| \leq \left(\left|\alpha - \frac{p_1}{q_1}\right|^{k-m} + \left|\alpha - \frac{p_2}{q_2}\right|\right) c_4^{\ell}$$

Pedriremos $\ell = \left\lfloor \frac{(1+\delta)kd}{2} \right\rfloor$. Si $\delta < 1 \Rightarrow \ell \leq \frac{(1+\delta)}{2} kd < kd$ (4)

Paso 3: No anulamiento de $R_m\left(\frac{p_1}{q_1}, \frac{p_2}{q_2}\right)$.

Lema 5: Dados $0<\delta<1$, $k \in \mathbb{N}$, $\ell = \left\lfloor \frac{(1+\delta)kd}{2} \right\rfloor$

$\exists\, C_5 = C_5(\alpha, \delta)$ tq $\forall \frac{p_1}{q_1}, \frac{p_2}{q_2} \in \mathbb{Q}$, $(p_1, q_1) = 1$ $q_1 > 0$, $q_2 > 0$

$\exists\, m \in \mathbb{N}$, $0 \leq m \leq \left\lfloor \frac{k C_5}{\log q_1} \right\rfloor + 2$ tq $R_m\left(\frac{p_1}{q_1}, \frac{p_2}{q_2}\right) \neq 0$.

Dem: Suponer $R_m\left(\frac{p_1}{q_1}, \frac{p_2}{q_2}\right) = 0$ $0 \leq m \leq t$

$$P^{(m)}\left(\frac{p_1}{q_1}\right) - \frac{p_2}{q_2} Q^{(m)}\left(\frac{p_1}{q_1}\right) = 0 \quad 0 \leq m, m' < t$$
$$P^{(m')}\left(\frac{p_1}{q_1}\right) - \frac{p_2}{q_2} Q^{(m')}\left(\frac{p_1}{q_1}\right) = 0$$

se hace aparte bajo las hipótesis del lema

$$\Rightarrow P^{(m)}\left(\frac{p_1}{q_1}\right) Q^{(m')}\left(\frac{p_1}{q_1}\right) - P^{(m')}\left(\frac{p_1}{q_1}\right) Q^{(m)}\left(\frac{p_1}{q_1}\right) = 0$$

$W(x) := P(x) Q'(x) - P'(x) Q(x) \Longrightarrow W^{(r)}\left(\frac{p_1}{q_1}\right) \quad 0 \leq r < t$

$\therefore \frac{p_1}{q_1}$ raíz de W de orden t

$\Rightarrow W(x) = (q_1 x - p_1)^t H(x)$, $H(x) \in \mathbb{Z}[x]$. $\boxed{\text{Suponer } W(x) \neq 0}$

Si w es el coef. más grande $|w| \leq C_6^{\ell} \leq (1 + C_6)^{\ell} < C_7^{k}$.

y $q_1^t \leq |w|$

$\Rightarrow$ logaritmo, $t \leq k \frac{\log(C_7)}{\log q_1}$ $\quad t \leq \left\lfloor \frac{k C_5}{\log q_1} \right\rfloor + 1$

Si $t = \left\lfloor \frac{k C_5}{\log q_1} \right\rfloor + 2$, $\exists$ el m del lema.

$$\left| R_m\left(\frac{p_1}{q_1}, \frac{p_2}{q_2}\right) \right| = \left| \sum_{i=0}^{\ell} \sum_{j=0}^{1} a_{ij} \binom{i}{m} p_1^{i-m} q_1^{-(i-m)} p_2^{j} q_2^{-j} \right|$$
$$= q_1^{-\ell} q_2^{-1} \left| \sum_{i=0}^{\ell} \sum_{j=0}^{1} a_{ij} \binom{i}{m} p_1^{i-m} q_1^{\ell+m-i} p_2^{j} q_2^{1-j} \right| \geq \frac{1}{q_1^{\ell}} \cdot \frac{1}{q_2}$$

$\boxed{\text{Paso 4}}$: juntar $\varepsilon > 0$, $\ell = \frac{d}{2} + 1 + \varepsilon$

más $\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^r}$ (*). Suponer que hay ∞ $\frac{p}{q} \in \mathbb{Q}$ que lo cumple $(p,q) = 1$, $q > 0$

i.e. luego hay infinitos q con $q \to \infty$. Sean $\frac{p_1}{q_1}, \frac{p_2}{q_2}$ $q_2 > q_1$ tal que (*). Sea $0 < \delta < \frac{1}{2}$, sea k natural tal que $q_1^k < q_2 < q_1^{k+1}$.

Tomemos que sea tan grande como para que $k \geq \frac{1}{\delta}$ tomemos $q_1 \geq e^{C_5/\delta}$
$\Rightarrow \log q_1 \geq \frac{C_5}{\delta} \Rightarrow 2 + \delta k \geq \left\lfloor \frac{C_5 k}{\log q_1} \right\rfloor + 2$.

$\exists\, m$, $0 \leq m \leq k\delta + 2$ tq $\left|R_m\left(\frac{p_1}{q_1}, \frac{p_2}{q_2}\right)\right| \geq q_1^{-\ell} q_2^{-1}$, $\ell = \left\lfloor \frac{(1+\delta)}{2} kd \right\rfloor$

i.e. $\left|R_m\left(\frac{p_1}{q_1}, \frac{p_2}{q_2}\right)\right| \geq q_1^{-\frac{1+\delta}{2}kd - k - 1}$

(Lema 4) $\left|R_m\left(\frac{p_1}{q_1}, \frac{p_2}{q_2}\right)\right| \leq \left(\left|\alpha - \frac{p_1}{q_1}\right|^{k-m} + \left|\alpha - \frac{p_2}{q_2}\right|\right) C^\ell \leq \left(q_1^{-r(k-m)} + q_2^{-r}\right) C^\ell$

$\ldots \left|R_m\left(\frac{p_1}{q_1}, \frac{p_2}{q_2}\right)\right| \leq \left(q_1^{-r(k(1-\delta)-2)} + q_1^{-rk}\right) C_1^\ell$
$\leq 2\, q_1^{-r(k(1-\delta)-2)} C_8^{kd} \leq q_1^{-r(k(1-\delta)-2)} C_9^k$

Pedir adicionalmente $q_1 > C_9^{\frac{1}{\delta}} \Rightarrow \downarrow \leq q_1^{-r(k(1-\delta)-2) + \delta k}$

$\therefore$ $-\frac{(1+\delta)}{2} kd - k - 1 \leq -r(k(1-\delta)-2) + \delta k$

$\Rightarrow 1 + 2r \geq rk - k\delta r - \delta k - k - \frac{(1+\delta)}{2} kd$

$\delta(1+2r) \geq \frac{1+2r}{k} \geq r - \delta r - \delta - 1 - \frac{(1+\delta)}{2} d$

$\Rightarrow r \leq \frac{d}{2} + 1 + \delta\underbrace{\left(1 + 2\ell + \ell + 1 + \frac{d}{2}\right)}_{v}$

$\Rightarrow$ Sea $\delta = \min\left\{\frac{1}{4}, \frac{\varepsilon}{2v}\right\}$

$\therefore r \leq \frac{d}{2} + 1 + \frac{\varepsilon}{2} < \underbrace{\frac{d}{2} + 1 + \varepsilon}_{= \ell}$ ■