# Alturas (Héctor Pastén, 30-Agosto)



- $K = \#$ field, $M_K = \{\text{places}\} = M_K^0 \cup M_K^\infty$
- $\mathfrak{p} \subset \mathcal{O}_K$ prime, $|x|_\mathfrak{p} = (N_\mathfrak{p})^{-v_\mathfrak{p}(x)}$ $\quad N_\mathfrak{p} = \# \mathcal{O}_K/\mathfrak{p}$

- $\sigma : K \to \mathbb{C}$, $|x|_\sigma = |\sigma(x)|$
- $v \in M_K$,

$$\|x\|_v = \begin{cases} |x|_\mathfrak{p} & , v = \mathfrak{p} \\ |x|_\sigma & , v = \{\sigma\} \text{ real} \\ |x|_\sigma^2 & , v = \{\sigma, \bar\sigma\} \text{ nonreal} \end{cases}$$

Multiplicative height on $\mathbb{P}^n(K)$:

$$H_K([x_0 : \dots : x_n]) = \prod_v \max_{0 \le j \le m} \|x_j\|_v \ge 1 \qquad h_K = \log H_K \ge 0$$

well-defined: By prod. formula: $\forall x \in K^*$, $\prod_{v \in K} \|x\|_v = 1$.

Ex).- $K = \mathbb{Q}$, $H_\mathbb{Q}([x]) = \max |x_j|$, $\gcd(x_0, \dots, x_n) = 1$.

- Normalized log Height; $h = \frac{1}{[K:\mathbb{Q}]} h_K : \mathbb{P}^n(\bar{\mathbb{Q}}) \longrightarrow \mathbb{R}_{\ge 0}$.

Thm (Northcott): Let $d \ge 1$, $B > 0$. The set $\{P \in \mathbb{P}^n(\bar{\mathbb{Q}}) : [\mathbb{Q}(P):\mathbb{Q}] \le d \;\&\; h(P) \le B\}$
$\Rightarrow$ This set is finite.

Dem: ① over $K = \mathbb{Q}$ clear ($d = 1$)
② general: reduce to $n = 1$.
Idea ($n = 1$): Compare $H_K(\alpha)$ to the size of coefficients of min poly of $\alpha$. ∎

Thm (Weil's height Machine) Let $K$ be # field. For each smooth proj variety $X|_K$, and each line sheaf $\mathcal{L}$ on $X$ over $K$ there is a function

$$h_{X,\mathcal{L}} : X(\bar K) \longrightarrow \mathbb{R} \quad \text{satisfying.}$$

Reference heights: HINDRY-SILVERMAN

(1) [Functoriality]: $\forall\, F: X_1 \longrightarrow X_2$ over $K$, $\mathcal{L}$ over $X_2$.
we have $h_{X_1, F^*\mathcal{L}} = h_{X_2,\mathcal{L}} \circ F + O(1)$.

(2) [Normalization]: $h_{\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)} = h + O(1)$ appears in rel. to ~~depends on~~ the different

(3) [additive]: $\forall\, \mathcal{L}, \mathcal{F}$ on $X$: $h_{X, \mathcal{L}\otimes\mathcal{F}^\vee} = h_{X,\mathcal{L}} - h_{X,\mathcal{F}} + O(1)$

(4) $\mathcal{L}_1 \simeq \mathcal{L}_2 \Rightarrow h_{X,\mathcal{L}_1} = h_{X,\mathcal{L}_2} + O(1)$
[linear equiv.]

(5) [effective]: $D \geqslant 0$ effective $\Rightarrow h_{X,\mathcal{O}_X(D)} \geqslant$ cte away from base locus of $D$.

(6) [Northcott]: $\mathcal{A}$ ample line sheaf $\Rightarrow h_{X,\mathcal{A}}$ has the Northcott property.

Furthermore: $h_{X,\mathcal{L}}$ is unique up to $+\,O(1)$.

Trick: Let $\mathcal{L}$ be line sheaf $\exists\, \mathcal{A}, \mathcal{B}$ very ample st $\mathcal{L} \simeq \mathcal{A} \otimes \mathcal{B}^\vee$.

Notation: $h_{X,D} := h_{X,\mathcal{O}(D)}$

Thm: (Néron-Tate) $K = \#$field, $X/_K$ smooth proj. variety, $D \in \mathrm{Div}(X/_K)$,
$\phi: X \twoheadrightarrow X$ over $K$. Suppose $\phi^*(D) \sim MD$ with $M > 1$.

Then $\exists!\ \hat{h}_{X,D,\phi} : X(\overline{K}) \longrightarrow \mathbb{R}$ such that
(1) $\hat{h}_{X,D,\phi} = h_{X,D} + O(1)$
(2) $\hat{h}_{X,D,\phi} \circ \phi = M\, \hat{h}_{X,D,\phi}$

Furthermore $\hat{h}_{X,D,\phi}$ is given by:

$$\hat{h}_{X,D,\phi}(P) = \lim_{n\to\infty} \frac{1}{M^n} h_{X,D}(\phi^{(n)}(P))$$

Pf. Use the formula, $M>d$, and check Cauchy ■

§ Abelian varieties

$A|_K$, $P_0 \in A(K)$, $i: A \to A$ inverse, $\sigma: A\times A \to A$ addition.
$\forall n \in \mathbb{Z}$, $[n]: A \to A$ étale of degree $n^{2\dim(A)}$

Thm (Mumford) $D$ divisor on $A|_K$, $n \in \mathbb{Z}\setminus\{0\}$. Then $[n]^*D \sim \left(\frac{n^2+n}{2}\right)D + \left(\frac{n^2-n}{2}\right)[-1]^*D$.

Def: $D$ is symmetric if $[-1]^*D \sim D$.
$D$ is antisymmetric if $[-1]^*D \sim -D$.

- $D$ sym $\Rightarrow$ $[n]^*D \sim n^2 D$

Def: $\delta: A\times A \to A$ difference. $A\times A \xrightarrow{\pi_i} A$ projections.

Thm (Mumford): (1) $D$ is antisym $\Rightarrow$ $\sigma^*D \sim \pi_1^*D + \pi_2^*D$
(2) $D$ is sym $\Rightarrow$ $\sigma^*D + \delta^*D \sim 2(\pi_1^*D + \pi_2^*D)$

Thm: let $D$ be divisor on $A|_K$, assume $D$ is symm. Then there is a unique canonical $\hat{h}_{A,D}: A(\bar{K}) \to \mathbb{R}$ satisfying
(1) $\hat{h}_{A,D} = h_{A,D} + O(1)$
(2) $\hat{h}_{A,D}\circ[2] = 4\hat{h}_{A,D}$

Furthermore:
(3) $\forall m$, $\hat{h}_{A,D}\circ[m] = m^2 \hat{h}_{A,D}$.
(4) $\hat{h}_{A,D}(P+Q) + \hat{h}_{A,D}(P-Q) = 2(\hat{h}_{A,D}(P) + \hat{h}_{A,D}(Q))$
(5) $\langle P,Q\rangle_{A,D} := \frac{1}{2}\{\hat{h}_{A,D}(P+Q) - \hat{h}_{A,D}(P) - \hat{h}_{A,D}(Q)\}$ is $\mathbb{Z}$-bilinear
(6) $D$ ample $\Rightarrow \{P\in A(\bar{K}) ; \hat{h}_{A,D}(P)=0\} = A(\bar{K})_{tor}$.

Proof: $\hat{h}_{A,D} = \hat{h}_{A,D,[2]}$ $D$ sym $\Rightarrow [2]^*D \sim 4D \Rightarrow (1),(2)$. Etc... ■

$\boxed{\text{Paso 4}}$: juntar $\varepsilon > 0$, $\ell = \frac{d}{2} + 1 + \varepsilon$ 5

mira $\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^{r}}$. Suponer que hay $\infty$ $\frac{p}{q} \in \mathbb{Q}$ que lo cumple
(*) $(p,q) = 1$, $q > 0$

i.e. luego hay infinitos $q$ con $q \to \infty$. Sean $\frac{p_1}{q_1}, \frac{p_2}{q_2}$ $q_2 > q_1$ tal que (*). Sea $0 < \delta < \frac{1}{2}$, sea $k$ natural tal que
$q_1^{k} < q_2 < q_1^{k+1}$.

Tomemos que sea tan grande como para que $k \geq \frac{1}{\delta}$ ~~tomemos~~ $q_1 \geq e^{C_5/\delta}$
$\Rightarrow \log q_1 \geq \frac{C_5}{\delta} \Rightarrow 2 + \delta k \geq \left\lfloor \frac{C_5 k}{\log q_1} \right\rfloor + 2$.

$\exists\, m$, $0 \leq m \leq k\delta + 2$ tq $\left|R_m\left(\frac{p_1}{q_1}, \frac{p_2}{q_2}\right)\right| \geq q_1^{-\ell} q_2^{-1}$, $\ell = \left\lfloor \frac{(1+\delta)}{2} kd \right\rfloor$

i.e. $\left|R_m\left(\frac{p_1}{q_1}, \frac{p_2}{q_2}\right)\right| \geq q_1^{-\frac{1+\delta}{2}kd - k - 1}$

(Lema 4) $\left|R_m\left(\frac{p_1}{q_1}, \frac{p_2}{q_2}\right)\right| \leq \left(\left|\alpha - \frac{p_1}{q_1}\right|^{k-m} + \left|\alpha - \frac{p_2}{q_2}\right|\right) C^{\ell} \leq \left(q_1^{-r(k-m)} + q_2^{-r}\right) C^{\ell}$

... $\left|R_m\left(\frac{p_1}{q_1}, \frac{p_2}{q_2}\right)\right| \leq \left(q_1^{-r(k(1-\delta)-2)} + q_1^{-rk}\right) C_1^{\ell}$
$\leq 2\, q_1^{-r(k(1-\delta)-2)} C_8^{kd} \leq q_1^{-r(k(1-\delta)-2)} C_9^{k}$

Pedir adicionalmente $q_1 > C_9^{\frac{1}{\delta}} \Rightarrow \ \leq q_1^{-r(k(1-\delta)-2) + \delta k}$

$\therefore\ -\frac{(1+\delta)}{2} kd - k - 1 \leq -r(k(1-\delta) - 2) + \delta k$

$\Rightarrow 1 + 2r \geq rk - k\delta r - \delta k - k - \frac{(1+\delta)}{2} kd$

$\delta(1 + 2r) \geq \frac{1+2r}{k} \geq r - \delta r - \delta - 1 - \frac{(1+\delta)}{2} d$

$\Rightarrow r \leq \frac{d}{2} + 1 - \delta\underbrace{\left(1 + 2\ell + \ell + 1 + \frac{d}{2}\right)}_{v}$

$\Rightarrow$ Sea $\delta = \min\left\{\frac{1}{4}, \frac{\varepsilon}{2v}\right\}$

$\therefore\ r \leq \frac{d}{2} + 1 + \frac{\varepsilon}{2} < \frac{d}{2} + 1 + \varepsilon = \ell$

$\blacksquare$