→ Cálculos concretos de Bauer ("Seshadri constants on alg. surf." 1999)

Teo: $X \subset \mathbb{P}^N$ smooth irred. surface y considerar $L = \mathcal{O}_X(1)$. $\varepsilon(X) := \inf_{x \in X} (\underbrace{\varepsilon(X;x)}_{:= \varepsilon(L;x)})$.

(a) $\varepsilon(X) = 1 \iff X$ contiene una recta.
En efecto, $\{x \in X : \varepsilon(X;x) = 1\} =$ rectas $\subset X$.

(b) Sea $d \geq 4$ y $S_{d,N} = \{X : X$ grado d sin rectas$\}$
$\Rightarrow \min\{\varepsilon(X) : X \in S_{d,N}\} = \frac{d}{d-1}$.

(c) $X \in S_{d,N}$ y $x \in X$ tal que $1 < \varepsilon(X,x) < 2$. Entonces
$\varepsilon(X,x) = \frac{a}{b}$ con $3 \leq a \leq d$ y $a/2 < b < a$.

(d) Todos los números a/b con $3 \leq a \leq d$ y $\frac{a}{2} < b < a$ aparecen como $\varepsilon(X,x)$ para $X \subset \mathbb{P}^3$ de grado d.

Deformations & Seshadri constants. <u>Subhendu</u>.

Let $f : X \to B$ be a family of polarized surfaces of deg d, $\mathcal{L}$ invertible on X
1. B = Noetherian scheme
2. f = flat
3. $\forall t \in B$, $\mathcal{L}|_{X_t}$ is ample on X_t, $\mathcal{L}|_{X_t}^2 = d$.
4. X_t = smooth proj. surface

Set $\Sigma = \{\varepsilon(\mathcal{L}_t; x_t) : t \in B, x_t \in X_t\}$

<u>Thm 1</u> (Oguiso) For each $\alpha \in \mathbb{R}$, $0 < \alpha < \sqrt{d}$
$\Sigma \cap (0, \alpha]$ is finite.

<u>Cor</u>: Let (X,L) be polari. surf. of deg d, and let $\varepsilon(L) = \inf_{x \in X} \varepsilon(L,x)$. If $\varepsilon(L) < \sqrt{d} \to \exists\, x \in X$, $\exists\, C \subseteq X$ s.t. $\varepsilon(L) = \varepsilon(L,x) = \frac{L \cdot C}{m_x(C)}$

in general $\varepsilon(L) = \min_{x \in X} \varepsilon(L,x)$.

Thm 2 (Oguiso) If $(\mathcal{X} \xrightarrow{f} B, \mathcal{L})$ is as above and
$\mathcal{X}(a) = \{ x_t \,/\, t \in B,\ x_t \in X_t,\ \varepsilon(\mathcal{L}_t, x_t) \leq a \}$
$\Rightarrow \mathcal{X}(a)$ is Zariski closed in $\mathcal{X}$.

Cor: 1. Fix $t \in B \Rightarrow$ the function $\varepsilon(x) := \varepsilon(\mathcal{L}_t, x)$
$x \in X_t$ is lower semi-cont. in Zar. topology of X_t
2. The function $\varepsilon(t) := \varepsilon(\mathcal{L}_t)$ is lower semi cont on B.

Lemma 1: (X, L) polarized surface of deg d, and $\varepsilon(L, x) < \sqrt{d} \Rightarrow \exists\, C \subseteq X$ curve
$x \in C$ s.t. $\varepsilon(L, x) = (L \cdot C)/m_x(C)$.

Lemma 2: $(X \xrightarrow{f} B, \mathcal{L})$ = pol. family of degree d. If $a \in \mathbb{Q}_{>0}$, $a < \sqrt{d}$
$\Rightarrow \exists\, B = B(a) \in \mathbb{N}$ st. $(\mathcal{L}_t \cdot C_t) \leq B \quad \forall t \in \mathcal{B}, \forall x_t \in X_t$
and $\forall$ curves $\underset{x_t}{C_t} \subseteq X_t$ s.t. $\mathcal{L}_t \cdot C_t / m_{x_t}(C) < a$.

Lemma 3: Let $\mathcal{H}'$ be the relative Hilbert scheme of 1 dim'l subschemes of the family $(\mathcal{X} \xrightarrow{f} B)$. Let $\mathcal{H}(B) \subseteq \mathcal{H}'$ be the subscheme corresp. to curves $C_t \subseteq X_t$ st. $C_t \mathcal{L}_t \leq B$. Then $\mathcal{H}(B)$ has finitely many irred. comps.

Proof: Given a poly. $P \in \mathbb{Q}[n]$, the subscheme $\mathcal{H}'_p \subseteq \mathcal{H}'$ of curves with Hilbert poly P (wrt $\mathcal{L}$) is proj. $/_B \Rightarrow \mathcal{H}'_p$ has finitely many comps.
Suffices to show that $\mathcal{L}_t \cdot C_t < B \Rightarrow C_t$ can only have finitely many Hilbert polyn.
RR: $\chi(C_t, \mathcal{L}_t^{\otimes n}) = n(C_t \cdot \mathcal{L}_t) + 1 - g$
$\Rightarrow$ suffices to show finitely many g.
[Hart IV 6.4] If $C \subseteq \mathbb{P}^3$ has degree $k \Rightarrow g(C) < \frac{k^2}{4}$.

Proof Thm 1: Easy (Oguiso)

<u>proof of Thm 2</u>: Fix $a \in \mathbb{Q}$, $0 < a < \sqrt{d}$, and let $x_t \in \mathcal{X}(a)$. since $a < \sqrt{d}$ $\exists$ curve $C_t \subseteq X_t$ s.t. $\mathcal{E}(\mathcal{L}_t, x_t) = \mathcal{L}_t \cdot C_t / m_{x_t}(C_t)$.

$\mathcal{H}'$ the relative Hilbert scheme of "curves" of $\mathcal{X} \xrightarrow{f} \mathcal{B}'$, set $K(a) \subseteq \mathcal{X} \times_{\mathcal{B}} \mathcal{H}'$ the set consisting of all pairs $[x_t \in C_t]$ s.t. $\mathcal{E}(\mathcal{L}_t, x_t) = \mathcal{L}_t \cdot C_t / m_{x_t}(C_t) \leq a$

Let $\mathcal{H}(a) = \overline{K(a)}$. Lemma 2 $\Rightarrow$ $\exists B \in \mathbb{N}$ s.t. $(\mathcal{L}_t \cdot C_t) \leq B$ $\forall [x_t \in \underbrace{C_t}_{\text{integral curves}}] \in K(a)$. Lemma 3 $\Rightarrow$ $\mathcal{H}(a)$ has finitely many compons.

Let $\mathcal{H}_a = \bigcup_{i=1} \mathcal{H}(a)_i$, $\mathcal{H}(a)_i$ = Comps, $K(a_i) = K(a) \cap \mathcal{H}(a)_i$ so $\overline{K(a_i)} = \mathcal{H}(a)_i$.

Tech fact [The relative Cartier div functor of $\mathcal{X} \xrightarrow{f} \mathcal{B}$ is closed, so if $[x_0 \in C_0] \in \mathcal{H}(a)_i$, then C_0 is Cartier, and so has no embedded comp.]

Let $\mathcal{U} \subseteq \mathcal{H}(a)_i$ be an open nbhd of $[x_0 \in C_0] \in \mathcal{H}(a_i)$)

...

Sukhendu II

<u>Thm (Ein-Laz)</u>: (X, L) = smooth proj surface, L ample. Then for all except countably many pts $x \in X$, $\mathcal{E}(L, x) \geq 1$. If $c_1(L)^2 > 1$ $\Rightarrow$ exceptional set is finite.

<u>Main idea</u>: The set of pairs (C, x), $x \in C$ and $\frac{C \cdot L}{\text{mult}_x(C)} < 1$ is param by countably many irred varieties. Show that each such is discrete.

Let $C_0 \subset \mathcal{C} \subseteq X \times \Delta$ family over Δ = disc with parameter t.

$$\begin{array}{ccc} C_0 & \subset & \mathcal{C} \\ \downarrow & x \nearrow & \downarrow \\ 0 & \in & \Delta \end{array}$$

$\Rightarrow$ $\exists$ Kodaira-Spencer map $\rho : T_0\Delta \longrightarrow H^0(C_0, N_{C_0/X})$, $N_{C_0/X} = \mathcal{O}_{C_0}(C_0)$

Lemma: Assume $m_{x_t}(C_t) \geqslant m$, $\forall t \in \Delta$

Then, $\rho(\frac{d}{dt}) \in H^0(C_0, N)$ vanishes to order $\geqslant m-1$ at x_0.

Pf.) local computation: Assume $C_0 \subseteq \mathcal{U} \subseteq \mathbb{C}^2$ with coord. (z,w).

Then $\mathcal{C} \subseteq \mathcal{U} \times \Delta$ is defined by a power series

$F(z,w,t) = f_t(z,w)$ where $C_t = \{f_t = 0\}$

Let $x_t = (a(t), b(t))$ a, b power series. Then $\phi_t(z,w) = F(z+a(t), w+b(t), t)$ defines curves with multiplicity $\geqslant m$ at $(0,0)$ $\forall t \in \Delta$.

Now, $\phi_t(z,w) = \sum_{i \geqslant 0} \phi_i(z,w)\, t^i$, where $\phi_i(z,w) \in (z,w)$

$$\underbrace{\phi_1(z,w)}_{\in (z,w)^m} = \underbrace{\frac{\partial f_0}{\partial z}(z,w,0) \cdot a'(0) + \frac{\partial f_0}{\partial w}(z,w,0)\, b'(0)}_{\in (z,w)^{m-1}} + \frac{\partial F}{\partial t}(z,w,0)$$

$$\Rightarrow \frac{\partial F}{\partial t}(z,w,0) \in (z,w)^{m-1}.$$

Cor: Let $\begin{matrix}\mathcal{C}\\ \downarrow\\ \Delta\end{matrix}$ be as above, $C = C_0$ integral. If $s = \rho(\frac{d}{dt}) \neq 0$

then $C^2 \geqslant m(m-1)$.

Pf.) Let $Y = Bl_x(X) \xrightarrow{f} X$ have excep. div E.

C' = proper transform of C $\Rightarrow$ $f^*(C) = C' + kE$ $\quad \overset{\text{mult}_x(C)}{\overset{\|}{k}} \geqslant m$

Lemma $\Rightarrow 0 \neq s \in H^0(\mathcal{O}_C(C) \otimes m_x^{m-1})$

$\Rightarrow \underset{0 \neq}{\exists}\, s' \in H^0(C', f^*(\mathcal{O}_C(C)) \otimes \mathcal{O}(-(m-1)E))$

$\Rightarrow \deg(f^*(\mathcal{O}_C(C))) \geqslant (m-1)E \cdot C' = k(m-1)$.

Also $\overset{\|}{C \cdot C} = \deg(f^*(\mathcal{O}_C(C)|_{C'})) \geqslant m(m-1)$.

Proof thm: $S_d = \left\{ (C,x) \;/\; x \in C = \text{integral curve},\ m_x(C) > C\cdot L,\ C\cdot L \le d \right\}$

From last thm, S_d is parametrized by finit. many red proj var.

$\Rightarrow S = \bigcup S_d = \left\{ (C,x) \;/\; \frac{C\cdot L}{\text{mult}_x(C)} < 1 \right\}$ is param by count. man.

<u>claim</u>: Each such family is discrete $\Rightarrow$ thm.

<u>Suppose not</u>: Then $\exists$ family of integral curves $x \in \begin{matrix}\mathcal{C} \subseteq X\times\Delta \\ \downarrow \\ \Delta = \text{disc}\end{matrix}$

where $\underset{=m}{\text{mult}_{x_t}(C_t)} > C_t\cdot L \quad \forall t$

As C_t is integral, $m_y(C_t) = 1$ for all but finitely many points

$\Rightarrow$ If (C_t, x_t) moves $\Rightarrow$ so does C_t

Pick t, s.t. $T_{x_t}\Delta \longrightarrow H^0(C_t, N_{C_t/X})$ is nonzero.

Then, $C_t^2 > m(m-1)$ Hodge index thm $\Rightarrow (C_t^2)(L^2) \le (C_t\cdot L)^2$

$\therefore\ m(m-1) \le (C_t^2)(L^2) \le (C_t\cdot L)^2 \le (m-1)^2$

Robert.

Def $X = Bl_{s\ \text{pts gen}}(\mathbb{P}^2) \xrightarrow{\sigma} \mathbb{P}^2 \quad E_1,\dots,E_s$ div excep.

$H = \sigma^*(\mathcal{O}_{\mathbb{P}^2}(1))$ $d, m_1, \dots, m_s$ enteros $\mathcal{L} = dH - \sum m_i E_i$

$\mathcal{L}$ <u>especial</u> si $h^0(\mathcal{L}) > 0,\ h^1(\mathcal{L}) > 0$

($\mathcal{L}$ no especial si $h^0(\mathcal{L}) = \max\left\{ \binom{d+2}{2} - \sum \binom{m_i+1}{2}, 0 \right\}$)

$\mathbb{E} = \sum E_i$.

Conjetura SHGH : Suponer $d, m_1, \ldots, m_s \geq -1$ con $d \geq m_i + m_j + m_k$ $\forall i, j, k$ distintos. Entonces $\mathcal{L}$ es no especial.

obs/ Cierto para $s \leq 9$.
obs/ SHGH $\Rightarrow$ Nagata ($H - \frac{1}{\sqrt{s}} E$ es nef).

Teo : $s \geq 9$ tq SHGH es verdad. Entonces o bien

(1) $\exists L \in \mathrm{Pic}(X_s)$ y $x \in X_s$ tq $\varepsilon(L; x)$ es irracional para x muy general

o (2) SHGH es falso para $s+1$.

Cor : Si todas las const. de Seshadri son racionales para $s = 9$
$\Rightarrow$ SHGH es falso para $s = 10$.

Dem : Supongamos SHGH es cierto para s.
$\Rightarrow$ Nagata es cierto para s.
Sea $\frac{1}{\sqrt{s+1}} < \delta < \frac{1}{\sqrt{s}}$ racional.

$\Rightarrow \tilde{\mathcal{L}} = H - \delta E$ es amplio. Supongamos que (1) $\varepsilon(\tilde{\mathcal{L}}, 1) = \max_x \{\varepsilon(\tilde{\mathcal{L}}, x)\} \in \mathbb{Q}$.
(2) $\varepsilon(\tilde{\mathcal{L}}, 1) < \sqrt{\tilde{\mathcal{L}}^2} \leftarrow$ irracional.

Lema : Para un punto general $P \in X$, $\exists \Gamma \subseteq \mathbb{P}^2$ divisor de grado γ con $M = \mathrm{mult}_{p_1} \Gamma = \ldots = \mathrm{mult}_{p_s} \Gamma$ y $m = \mathrm{mult}_p \Gamma$ cuyo transf. propia en X computa $\varepsilon(\tilde{\mathcal{L}}, P)$.

$$\varepsilon(\tilde{\mathcal{L}}, P) = \frac{\tilde{\mathcal{L}} \cdot \tilde{\Gamma}}{m} = \frac{\gamma - \delta M s}{m} < \sqrt{\tilde{\mathcal{L}}^2} = \sqrt{1 - s\delta^2}$$

$$\Rightarrow \gamma < m\sqrt{1 - s\delta^2} + \delta s \cdot M.$$

Supongamos que SHGH es cierto para $s+1$

$\Rightarrow$ Nagata es cierto para $s+1$

$$\left(H - \frac{1}{\sqrt{s+1}}\left(\mathbb{E} + E_{s+1}\right)\right)\cdot\tilde{\Gamma} \geq 0 \qquad \Rightarrow \quad r - \frac{1}{\sqrt{s+1}}(sM+m)$$

$$\Rightarrow \quad \frac{r}{sM+m} \geq \frac{1}{\sqrt{s+1}}$$

$$\therefore \quad r \geq 2M + m.$$

$\therefore$ $rH - M\mathbb{E} - mE_{s+1}$ satisface SHGH.

$\Rightarrow$ no especial y $\tilde{\Gamma} \in |rA - M\mathbb{E} - mE_{s+1}|$

$\Rightarrow$ $h^0(\tilde{\Gamma}) = r(r+3) - sM(M+1) - m(m+1) + 1 \geq 1$.

pero $r < m\sqrt{1-s\delta^2} + \delta sM$ es contradictorio $\blacksquare$