Subhendu
10/June/2006

Degenerations of linear systems.

$\mathcal{L}_d(m^n) = \{ C \subseteq \mathbb{P}^2 : \deg C = d,\ C$ has mult. at least m at n gen. points $\}$

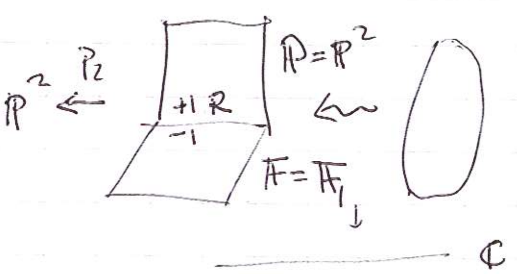
virtual dimension of $\mathcal{L}_d(m^n)$, $v = \frac{d(d+3)}{2} - n\frac{m(m+1)}{2}$.

Conj (Nagata): $\mathcal{L}_d(m^n) = \emptyset$ if $d < m\sqrt{n}$, $n > 9$.
Conj (Harb-Hirsch): $\mathcal{L}_d(m^n)$ has expected dim if $n > 9$.

Thm: HH is true if $n = k^2$.

Idea: Degenerate $\mathbb{P}^2$ to a reducible surface $\mathbb{P} \cup \mathbb{F}$ and distribute the points suitably. Use induction and semicontinuity.

Consider the 3-fold $\mathbb{P}^2 \times \mathbb{C}$, $R \subseteq \mathbb{P}^2 \times \{0\}$ be line, Set. $\mathcal{X} = Bl_R(\mathbb{P}^2 \times \mathbb{C})$
Let $\pi : \mathcal{X} \to \mathbb{C}$ be the second proj



The $n = k^2$ general points are made to degenerate to $(k-1)^2$ points on $\mathbb{P}$, and $(2k-1)$ pts on $\mathbb{P}$.

$$\mathcal{L} = p_2^*\mathcal{O}(d) \otimes \mathcal{O}_{\mathcal{X}}(m\mathbb{P}), \quad \mathcal{L}|_{X_t} \simeq \mathcal{O}_{\mathbb{P}^2}(d),\ t \neq 0.$$

$$\mathcal{L}|_{\mathbb{P}} \cong \mathcal{O}_{\mathbb{P}}(d-m) \quad \mathcal{L}|_{\mathbb{F}} \simeq \mathcal{O}_{\mathbb{F}}(mR + dF) \simeq \mathcal{O}_{\mathbb{F}}(dH - (d-m)R)$$

Let $L_{\mathbb{P}}$ be the linear subsystem of the linear system of $\mathcal{L}|_{\mathbb{P}}$ obtained by imposing the mult. condts on the $(k-1)^2$ points of $\mathbb{P}$. Define $L_{\mathbb{F}}$ similarly.

Let L_0 be the limit linear system on $\mathcal{X}_0$ (with $n = k^2$ mult conds)

Then, $H^0(\mathcal{X}_0, \mathcal{L}_0) = H^0(\mathbb{P}, \mathcal{L}_{\mathbb{P}}) \times_{H^0(R, \mathcal{L}|_R)} H^0(\mathbb{F}, \mathcal{L}_{\mathbb{F}})$.

Reduction : • $k \geq 4$
• $d = km + s$, some s, $0 \leq s \leq k-3$.

$l_{\mathbb{P}} = \dim L_{\mathbb{P}}$ $\quad l_{\mathbb{F}} = \dim L_{\mathbb{F}}$ $\quad \hat{l}_{\mathbb{P}} = \dim(\ker(H^0(\mathcal{L}_{\mathbb{P}}) \to H^0(\mathcal{L}|_R))) - 1$

$\hat{l}_{F} = \dim(\ker(H^0(\mathcal{L}_{\mathbb{F}}) \to H^0(\mathcal{L}|_R))) - 1$

$r_{\mathbb{P}} = \dim(\mathcal{L}_{\mathbb{P}}|_R) - 1 \qquad r_{\mathbb{F}} = \dim(\mathcal{L}_{\mathbb{F}}|_R) - 1$

$\mathcal{L}_{\mathbb{P}}, \mathcal{L}_{\mathbb{F}}, \mathcal{L}_0 \subset \mathcal{L}$

$l_{\mathbb{P}} - \hat{l}_{\mathbb{P}} - 1 = r_{\mathbb{P}}$

$l_{\mathbb{F}} - \hat{l}_{\mathbb{P}} - 1 = r_{\mathbb{F}}$.

Prop (*) Let $l_0 = \dim H^0(X_0, \mathcal{L}_0) - 1$. then

$$l_0 = \begin{cases} \hat{l}_{\mathbb{P}} + \hat{l}_{\mathbb{F}} + 1, & \text{if } r_{\mathbb{P}} + r_{\mathbb{F}} \leq d-m-1 \\ l_{\mathbb{P}} + l_{\mathbb{F}} - d + m, & \text{if } r_{\mathbb{P}} + r_{\mathbb{F}} \geq d-m-1. \end{cases}$$

Idea of proof : This relies on the result that for general points,

$H^0(\mathcal{L}_{\mathbb{P}}|_R) \cap H^0(\mathcal{L}_{\mathbb{F}}|_R)$ is a transversal interse.

Prop/- Let $V, W \subseteq H^0(\mathbb{P}^1, \mathcal{O}(b))$ be linear subspaces. Then,
$\exists g \in PGL(2, \mathbb{C}) = Aut(\mathbb{P}^1)$ st $V \pitchfork g(W)$ is transversal.

Strategy : $\dim \mathcal{L}_d(m^n) - 1 = e$ suffices to show that $l_0 = e$ by semicontinuity.

Exercise : let $C \subseteq \mathbb{P}^2$ be a curve of degree d with mult m_i at $p_i \in \mathbb{P}^2$ $i = 1, 2, 3)$ $(2d \geq m_1 + m_2 + m_3)$ then the Cremona transf based at p_1, p_2, p_3 transf C into a curve deg $2d - m_1 - m_2 - m_3$ with mult, at least $d - m_2 - m_3$ at q_1, $d - m_3 - m_1$ at q_2, $d - m_1 - m_2$ at q_3.

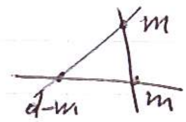
Regarding $\mathbb{F}$ as $Bl_p(\mathbb{P}^2)$, $L_{\mathbb{F}} = \mathcal{L}_d(d-m, m^{2k-1})$

Lemma 1 : $\forall k \geq 1$, $L_{\mathbb{F}}$ has the expected dim $\ell_{\mathbb{F}} = ms + m + s$

Pf : Induction on k

$k=1$: $\mathcal{L}_{m+s}(s, m)$ has expected dim.

$k>1$: Perform Cremona on

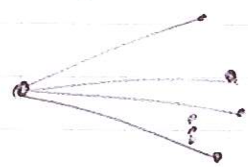


$\rightsquigarrow$ system of degree $d-m$, with mul $d-2m$, $2k-3$ of mult m $\rightsquigarrow$ reduce to 1 point.

Lemma 2 : $\forall k \geq 1$, $\hat{L}_{\mathbb{F}} = \phi$.

proof lr. Note $\hat{L}_{\mathbb{F}} = \mathcal{L}_d(d-m+1, m^{2k-1})$

Trick



$\Rightarrow \mathcal{L}_{d-2k+1}(d-m+1-2k+1, (m-1)^{2k-1})$

$\Rightarrow$ Cremona & done

$\therefore$ $r_{\mathbb{F}} = ms + m + s$

Next consider $L_{\mathbb{P}}$, $\hat{L}_{\mathbb{P}}$. Note $L_{\mathbb{P}} = \mathcal{L}_{d-m}(m^{(k-1)^2})$

$\hat{L}_{\mathbb{P}} = \mathcal{L}_{d-m-1}(m^{(k-1)^2})$

We use induction on $L_{\mathbb{P}}$, $\hat{L}_{\mathbb{P}}$ have expected dim.

Case 1 : $L_p = \phi$ ~ exercise.

Case 2 : $L_p \neq \phi$, $\hat{L}_{\mathbb{P}} = \phi$ $\Rightarrow$ $r_{\mathbb{P}} = \ell_p$ and so.

$r_{\mathbb{P}} + r_{\mathbb{F}} = \dots = r + (d-m)$.