Lecture #1 : $\boxed{\text{What are Stacks}}$?

(1) Fine moduli schemes are rare.

- Moduli spaces give a geometric answer to the question of classification. Morally, A moduli space X of some collection of objects $\mathcal{C}$ should be a space (variety, scheme, ... ?) where points should correspond to the objects in $\mathcal{C}$. This is not enough! Indeed, this only defines the underlying set structure, we want our answer to be geometric.

Def: A fine moduli scheme (*) $\mathcal{M}$ for a collection of objects $\mathcal{C}$ is a scheme $\mathcal{M}$ s.t. $\mathrm{Hom}_{Sch}(T, \mathcal{M}) \underset{\text{Natural}}{=} \{\text{families of objects in } \mathcal{C} \text{ parametrized by } T\}/\cong$ for all schemes T.

Ex: Subschemes of $\mathbb{P}^n$ ($\mathrm{Hilb}_{\mathbb{P}^n}$): let $\mathcal{C}$ be the collection of $X \underset{\text{closed}}{\hookrightarrow} \mathbb{P}^n$, & a family of subschemes of $\mathbb{P}^n/_T$ is

$$\begin{array}{ccc} X & \underset{f_X}{\overset{\text{closed}}{\hookrightarrow}} & \mathbb{P}^n_T \\ & {\scriptstyle \text{flat, finitely-presented}} \searrow & \downarrow \\ & & T \end{array}$$

, iso's are T-iso's respecting f_X & f_X'.

$$\begin{array}{ccc} X & \xrightarrow{\sim} & X' \\ \downarrow & \swarrow & \\ \mathbb{P}^n_T & & \end{array}$$

Def: A *moduli problem* is a functor $F : \underline{Sch} \to \underline{Set}$, & if $F \cong h_M$ for some $M \in Sch$, we say F is *representable* by M & that M is a fine moduli space.

Ex: Curves: $\mathcal{M}_g : \underline{Sch} \to \underline{Set}$ sends

$$T \longmapsto \left\{ \begin{array}{c} C \\ \downarrow \\ T \end{array} \begin{array}{l} \text{- proper} \\ \text{- flat} \\ \text{- finitely presented} \\ \text{- geometric fibers are smooth curves of genus } g \end{array} \right\} \Big/ \cong$$

$$\begin{tikzcd} C \arrow[rr, "\sim"] \arrow[dr] & & C' \arrow[dl] \\ & T & \end{tikzcd}$$ a T-morphism.

Ex: Vector bundles: $\underline{BGL_n} : \underline{Sch} \to \underline{Set}$ $\quad T \longmapsto \{\text{v.b.'s of rk } n \text{ on } T\}/\cong$.

$V \xrightarrow{\sim} V'$
$\mathcal{O}_X$-module map.

Question: Do these moduli problems have a fine moduli scheme?

$\boxed{\text{Thm}}$ (Grothendieck) $\underline{\mathrm{Hilb}_{\mathbb{P}^n}}$ is representable by a scheme & each of its connected components are proper.

Unfortunately, that is the only good news.

$\boxed{\text{Claim}}$: M_g is not representable by a scheme for $g \geq 0$.

Pf: Consider $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-1))$, the Hirzebruch surfaces are not all iso.

$$\begin{tikzcd} \mathbb{P}(\mathcal{O}\oplus\mathcal{O}(-1)) \arrow[d] \\ \mathbb{P}^1 \end{tikzcd}$$

which means if $\exists$ map $\mathbb{P}^1 \to M_0$ it must factor through a (closed) pt.

$$\begin{tikzcd} & \mathcal{U}_k \arrow[r] \arrow[d] & \mathcal{U} \arrow[d] \\ \mathbb{P}^1 \arrow[r] & \operatorname{Spec} k \arrow[r] & M \arrow[d] \\ \operatorname{Spec}\mathbb{C} \arrow[u] & & \operatorname{Spec}\mathbb{C} \end{tikzcd} \Rightarrow \quad \mathbb{P}(\mathcal{O}\oplus\mathcal{O}(-1)) \cong \mathcal{U}_k \Rightarrow \mathbb{C} \subseteq k \subseteq \mathbb{C} \Rightarrow \text{it must be a closed pt.}$$

hence $\mathbb{P}(\mathcal{O}\oplus\mathcal{O}(-1)) \cong \mathbb{P}^1 \times \mathbb{P}^1$ ↯

Why isn't $BGL_n : \underline{Sch} \to \underline{Set}$ representable by a scheme? Note that the functors representable by schemes have a nice property: they are $\underline{\text{sheaves}}$ on the Zariski topology, i.e. given $h_X : \underline{Sch} \to \underline{Set}$ & an open cover $T = \bigcup_{i \in I} T_i$ we have

$$\operatorname{Hom}(T, X) \hookrightarrow \prod_{i \in I} \operatorname{Hom}(T_i, X) \underset{p_2}{\overset{p_1}{\rightrightarrows}} \prod_{(i,j) \in I^2} \operatorname{Hom}(T_i \cap T_j, X)$$

This can be refined, the idea is to generalize the notion of an open set in topology so it makes sense to define a sheaf.

Def: A $\boxed{\text{site}}$ is a category $\mathcal{C}$ equipped w/ a set $\operatorname{Cov}(\mathcal{C})$ consisting of families of morphisms w/ fixed target $\{U_i \to U\}_{i \in I}$ called $\underline{\text{coverings}}$ (iso) which satisfy

1) (Iso's are covers) If $V \xrightarrow{\sim} U \Rightarrow \{V \xrightarrow{\sim} U\} \in Cov(\mathcal{C})$
2) (Covering compose) If $\{U_i \to U\}_{i\in I} \in Cov$ & $\forall i \in I$ $\exists$ $\{V_{ij} \to U_i\}_{i\in I} \in Cov(\mathcal{C}) \Rightarrow \{V_{ij} \to U\} \in Cov(\mathcal{C})$
3) (Coverings pullback) If $\{U_i \to U\} \in Cov(\mathcal{C})$ & $V \to U$ a map $\Rightarrow \{U_i \times_U V \to V\} \in Cov(\mathcal{C})$.

Eg. 1) Zar-top $\{U_i \to U\}$ is in $Cov((Sch/S)_{Zar})$ iff $U_i \to U$ is an open immersion.

Etale top: 2) $\{U_i \to U\}_{i\in I}$ is in $Cov((Sch/S)_{Et})$ iff $U_i \to U$ is étale & $\coprod_{i\in I} U_i \to U$ is surj.

Fppf-topology 3) $\{U_i \to U\}_{i\in I}$ is in $Cov((Sch/S)_{fppf})$ iff $U_i \to U$ is faithfully flat of locally f. presentation, jointly surjective

<u>Def</u>: A sheaf on a site $\mathcal{C}$ is a functor $F: \mathcal{C} \to \underline{Set}$ s.t. for any covering $\{U_i \to U\}_{i\in I}$ the sequence
$$F(U) \to \prod_{i\in I} F(U_i) \rightrightarrows \prod_{(i,j)\in I^2} F(U_i \times U_j)$$
is exact.

<u>Theorem</u>: If X is a S-scheme then $h_X: (Sch/S) \to \underline{Set}$ is a sheaf.
<u>Ex</u>: $BGL_n \neq h_X$ for a scheme X b/c any v.b. on T, V say is trivial on an open cover!

② <u>Stacks remember isomorphisms</u>
The problem is that we are making too many identifications w/ out remembering <u>how</u> they are identified.

<u>Moral</u>: Consider functors $F: \underline{Sch} \to \underline{Grp}$ (or categories instead (all maps are iso's))

So $BGL_n: \underline{Sch} \to \underline{Cat}$ where $T \mapsto \{$Vector bundles on $T\}$ category w/ morphisms are isomorphisms

where maps & objects should <u>glue</u>.

<u>Def</u>: Fix a functor from a site $\mathcal{X}: \mathcal{C} \to \underline{Grp}$, if $\{U_i \to U\}_{i\in I}$ is a covering (replace by $\underbrace{\coprod_{i\in I} U_i}_{U'} \to U$) define the cat. of <u>descent data</u> to be $\mathcal{X}(U' \to U)$
$$\{(x', \varphi) \mid x' \in \mathcal{X}(U'),\ p_1^* x' \xrightarrow[\sim]{\varphi} p_2^* x'$$ where $U'' = U' \times_U U' \overset{p_1}{\underset{p_2}{\rightrightarrows}} U'$
& s.t. $p_{13}^*\varphi = p_{23}^*\varphi \circ p_{12}^*\varphi$ where $U''' = U' \times_U U' \times_U U' \overset{p_{13}}{\underset{p_{23}}{\overset{p_{12}}{\rightrightarrows}}} U' \times_U U' = U''$ }.

A map $(x', \varphi) \to (y', \phi)$ is $x' \xrightarrow{f} y'$ s.t.

$$\begin{tikzcd} p_1^* x' \arrow[r, "p_1^* f"] \arrow[d, "\varphi"'] & p_1^* y' \arrow[d, "\phi"] \\ p_2^* x' \arrow[r, "p_2^* f"'] \arrow[ur, phantom, "\circlearrowright"] & p_2^* y' \end{tikzcd}$$

Note: There is a nat'l functor $\mathcal{X}(U) \to \mathcal{X}(U' \to U)$ b/c
$x \longmapsto (x, can)$ where can is defined by

$$\begin{tikzcd} U' \times_U U' \times_U U' \arrow[d, "p_{12}\ p_{23}\ p_{13}"] \\ U' \times_U U' \arrow[d, "p_1\ p_2"] \\ U' \arrow[d, "p"] \\ U \end{tikzcd}$$

$p \circ p_1 = p \circ p_2 \Rightarrow p_1^* p^* x \underset{can}{=} p_2^* p^* x$, also we have

$$\begin{tikzcd} p_{12}^* p_1^* p^* x \arrow[r, phantom, "\overset{p_1 p_{12} = p_1 p_{13}}{=}"] \arrow[d, "p_{12}^* can"'] & p_{13}^* p_1^* p^* x \arrow[r, "p_{13}^* can"] & p_{13}^* p_2^* p^* x \arrow[r, phantom, "\overset{p_2 \circ p_{13} = p_2 \circ p_{23}}{=}"] & p_{23}^* p_2^* p^* x \\ p_{12}^* p_2^* p^* x \arrow[rrr, equal, "b/c\ p_2 \circ p_{12} = p_1 \circ p_{23}"'] & & & p_{23}^* p_1^* p^* x \arrow[u, "p_{23}^* can"'] \end{tikzcd}$$

i.e. $p_{23}^* can \circ p_{12}^* can \overset{!}{=} p_{13}^* can$

Ex: For $BGL_n : (Sch/S)_{Zar} \to \underline{Grp}$ (vector bundles) let $\{U_i \to X\}$ be an open cover of X. The cat. of descent data $BGL_n(\{U_i \to X\})$ is equivalent to the cat.

$\{(V_i, (\varphi_{ij})) \mid V_i \in Vect(U_i),\ \varphi_{ij} : V_i|_{U_i \cap U_j} \xrightarrow{\sim} V_j|_{U_i \cap U_j}$ s.t. on $U_i \cap U_j \cap U_k$ $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}\}$

Def: A functor on a site $\mathcal{X} : (Sch/S)_* \to \underline{Cat}$ is a stack if
$* \in \{Zar, Ét, fppf\}$

1) For every S-scheme T & any $a, b \in \mathcal{X}(T)$ the functor

$\underline{Hom}(x, y) : (Sch/T)_* \to \underline{Set}$ is a sheaf.

2) For any covering $\{T_i \to T\}$ in $(Sch/S)_*$ every descent datum $(x, \varphi) \in \mathcal{X}(U' \to U)$ is effective (in the essential image of the natural functor $\mathcal{X}(U) \to \mathcal{X}(U' \to U)$)

Equivalently $\mathcal{X}(U) \to \mathcal{X}(U' \to U)$ is an equivalence of cats for every $U \in (Sch/S)_*$ & every cover $U' \to U$.

We will show that $\mathcal{M}_g$ & BGL_n are stacks.

We will need the following fundamental

Theorem. Let $S' \xrightarrow{p} S$ be a fppf + q.c. morphism of schemes then

1) $\underline{Hom}_{\mathcal{O}_X\text{-mod}}(\mathcal{F}, \mathcal{G}) \to S_{fppf}$ is a sheaf for $\mathcal{F}, \mathcal{G} \in \underline{Qcoh}_S$

2) $Qcoh(S) \xrightarrow{p^*} Qcoh(S' \to S)$ is an equivalence of categories

(In fact p^* fully faithful $\Leftrightarrow$ 1))

Cor $BGL_n = Vect : (Sch)_S \to \underline{Grp}$ sending $(T \to S) \mapsto \{V \mid V \text{ vect } / \text{ on } T\}$ is a stack.

$\underline{Pf}$: • $\underline{Hom}_{\mathcal{O}_T}(V, W) \to T_{fppf}$ is a sheaf b/c $V, W \in Qcoh_T$.

• $Vect(S) \to Vect(S' \xrightarrow[\text{fppf cover}]{} S)$ is an equivalence b/c if a q.coh $\tilde{M}$ is fppf locally a vector bundle! $\square$

Cor $\mathcal{M}_g$ $(g \neq 1)$ is a stack, $\mathcal{M}_1$ is $\underline{\text{not}}$.

$\underline{Pf}$: WTS that if
$$\begin{array}{ccc} C & & D \\ & \searrow \swarrow & \\ & S & \end{array}$$
are two objects of $\mathcal{M}_g(S)$ & S' is a covering then

$$\begin{array}{ccccc} C'' & \rightrightarrows & C' = C \times_S S' & \to & C \\ \downarrow & & \downarrow & & \downarrow \\ S'' & \rightrightarrows & S' & \to & S \end{array}$$

$$Hom_S(C, D) \to Hom(C', D|_{S'}) \rightrightarrows Hom(C'', D|_{S''})$$

(this follows b/c h_D is a sheaf: $h_D(C) \to h_D(C') \rightrightarrows h_D(C'')$ $\square$