Lecture #1 : $\boxed{\text{What are Stacks}}$?

① Fine moduli schemes are rare.

- Moduli spaces give a geometric answer to the question of classification. Morally, "A moduli space X of some collection of objects $\mathcal{C}$ should be a space (variety, scheme, ... ?) whose points should correspond to the objects in $\mathcal{C}$. This is not enough! Indeed, this only defines the underlying set structure, we want our answer to be geometric.

Def: A fine moduli scheme (*) $\mathcal{M}$ for a collection of objects $\mathcal{C}$ is a scheme $\mathcal{M}$ s.t.

$$\mathrm{Hom}_{Sch}(T, \mathcal{M}) \underset{\text{Natural}}{=} \left\{ \begin{array}{c} \text{families of objects in } \mathcal{C} \\ \text{parametrized by } T \end{array} \right\} \Big/ \cong$$

for all schemes T.

Ex: Subschemes of $\mathbb{P}^n$ ($\mathrm{Hilb}_{\mathbb{P}^n}$): let $\mathcal{C}$ be the collection of $X \underset{\text{closed}}{\hookrightarrow} \mathbb{P}^n$, & a family of subschemes of $\mathbb{P}^n/T$ is

$$\begin{tikzcd} X \arrow[r, hook, "\text{closed}", "f_X"'] \arrow[dr, "{\text{flat, finitely-presented}}"'] & \mathbb{P}^n_T \arrow[d] \\ & T \end{tikzcd}$$

, iso's are T-iso's respecting f_X & f_X'.

$$\begin{tikzcd} X \arrow[r, "\sim"] \arrow[d] & X' \arrow[dl] \\ \mathbb{P}^n_T & \end{tikzcd}$$

Def: A <u>moduli problem</u> is a <u>functor</u> $F: \underline{Sch} \to \underline{Set}$, & if $F \cong h_M$ for some $M \in Sch$, we say F is <u>representable</u> by M & that M is a fine moduli space.

Ex: <u>Curves</u>: $\underline{M_g}: \underline{Sch} \to \underline{Set}$ sends $T \longmapsto \left\{ \begin{array}{c} C \\ \downarrow \\ T \end{array} \right.$ - proper
- flat
- finitely presented
- geometric fibers are smooth curves of genus g $\Big\} / \cong$

$$\begin{tikzcd} C \arrow[rr, "\sim"] \arrow[dr] & & C' \arrow[ll, bend right] \arrow[dl] \\ & T & \end{tikzcd}$$ a T-morphism.

Ex: <u>Vector bundles</u>: $\underline{BGL_n}: \underline{Sch} \to \underline{Set}$ $\quad T \longmapsto \{$v.b.'s of rk $n\}/\cong$.

$\underbrace{V \xrightarrow{\sim} V'}$
$\mathcal{O}_X$-module map.

<u>Question</u>: Do these moduli problems have a <u>fine moduli scheme</u>?

$\boxed{\text{Thm}}$ (Grothendieck) <u>$\mathrm{Hilb}_{\mathbb{P}^n}$</u> is representable by a scheme & each of its connected components are proper.

Unfortunately, that is the only good news.

$\boxed{\text{Claim}}$: $\mathcal{M}_g$ is not representable by a scheme for $g \geq 0$.

Pf: Consider $\mathbb{P}(\mathcal{O}\oplus\mathcal{O}(-1))$, the Hirzebruch surfaces are not all iso.

$$\begin{array}{c}\mathbb{P}(\mathcal{O}\oplus\mathcal{O}(-1)) \\ \downarrow \\ \mathbb{P}^1\end{array}$$

which means if $\exists$ map $\mathbb{P}^1 \to \mathcal{M}_0$ it must factor through a (closed) pt.

$$\begin{tikzcd} & \mathcal{U}_k \arrow[r] \arrow[d] & \mathcal{U} \arrow[d] \\ \mathbb{P}^1 \arrow[r] & \operatorname{Spec} k \arrow[r] & \mathcal{M} \arrow[d] \\ \operatorname{Spec}\mathbb{C} \arrow[u] & & \operatorname{Spec}\mathbb{C} \end{tikzcd}$$

$\Rightarrow$ $\mathbb{P}(\mathcal{O}\oplus\mathcal{O}(-1)) \cong \mathcal{U}_k \Rightarrow \mathbb{C} \subseteq k \subseteq \mathbb{C} \Rightarrow$ it must be a closed pt.

hence $\mathbb{P}(\mathcal{O}\oplus\mathcal{O}(-1)) \cong \mathbb{P}^1 \times \mathbb{P}^1$ $\lightning$

Why isn't $BGL_n : \underline{Sch} \to \underline{Set}$ representable by a scheme? Note that the functors representable by schemes have a nice property: they are $\underline{\text{sheaves}}$ on the Zariski topology, i.e. given $h_X : \underline{Sch} \to \underline{Set}$ & an open cover $T = \bigcup_{i\in I} T_i$ we have

$$\operatorname{Hom}(T, X) \hookrightarrow \prod_{i\in I} \operatorname{Hom}(T_i, X) \overset{p_1}{\underset{p_2}{\rightrightarrows}} \prod_{(i,j)\in I^2} \operatorname{Hom}(T_i \cap T_j, X)$$

This can be refined, the idea is to generalize the notion of an open set in topology so it makes sense to define a sheaf.

Def: A $\boxed{\text{site}}$ is a category $\mathcal{C}$ equipped w/ a set $\operatorname{Cov}(\mathcal{C})$ consisting of families of morphisms w/ fixed target $\{U_i \to U\}_{i\in I}$ called $\underline{\text{coverings}}$ (Isos) which satisfy

1) (Iso's are covers) If $V \xrightarrow{\sim} U \Rightarrow \{V \xrightarrow{\sim} U\} \in Cov(\mathcal{C})$
2) (Covering compose) If $\{U_i \to U\}_{i\in I} \in Cov$ & $\forall i \in I$ $\exists \{V_{ij} \to U_i\} \in Cov(\mathcal{C}) \Rightarrow \{V_{ij} \to U\} \in Cov(\mathcal{C})$
3) (Coverings pullback) If $\{U_i \to U\} \in Cov(\mathcal{C})$ & $V \to U$ a map $\Rightarrow \{U_i \times_U V \to V\} \in Cov(\mathcal{C})$.

Eg. Zar-top 1) $\{U_i \to U\}$ is in $Cov((Sch/S)_{Zar})$ iff $U_i \to U$ is an open immersion.

Etale top: 2) $\{U_i \to U\}_{i\in I}$ is in $Cov((Sch/S)_{Et})$ iff $U_i \to U$ is étale & $\bigsqcup_{i\in I} U_i \to U$ is surj.

Fppf-topology 3) $\{U_i \to U\}_{i\in I}$ is in $Cov((Sch/S)_{fppf})$ iff $U_i \to U$ is faithfully flat of locally f. presentation jointly surjective

Def: A sheaf on a site $\mathcal{C}$ is a functor $F: \mathcal{C} \to \underline{Set}$ s.t. for any covering $\{U_i \to U\}_{i\in I}$ the sequence $F(U) \to \prod_{i\in I} F(U_i) \rightrightarrows \prod_{(i,j)\in I^2} F(U_i \times U_j)$ is exact.

Theorem: If X is a S-scheme then $h_X: (Sch/S) \to \underline{Set}$ is a sheaf.
Ex: $BGL_n \neq h_X$ for a scheme X b/c any v.b. on T, V say is trivial on an open cover!

② Stacks remember isomorphisms
The problem is that we are making too many identifications w/out remembering how they are identified.

Moral: consider functors $F: \underline{Sch} \to \underline{Grp}$ & categories instead. (all maps are iso's)

$\boxed{\text{Lecture 2}}$

Last time: Want moduli problems $F: Sch_S \to \underline{Set}$ to be rep'ble

b/c then F is amenable to geometric methods

$F(-) \cong h_X(-) = Hom(-, X)$. But h_X is a sheaf on the fppf site $(Sch_S)_{fppf}$ so this puts a restriction on the moduli problems F which can be represented by schemes e.g. if V is a non-trivial v.b. on S then $BGL_n: Sch_S \to Set$, $T \mapsto \{\text{v.b.'s of rank } n\}/\sim$ is not a sheaf b/c $[V], [\mathcal{O}_S^{\oplus n}] \in BGL_n(S)$ of rank n are locally but not globally isomorphic.

Thus enters the notion of a stack. Let $F: \underline{Sch}_S \to \underline{Grp}$ be a functor & let $S' \xrightarrow[p]{} S$ be a cover in a site; ↙think of $S' = \coprod_{i=1}^{m} U_i$ where $U_i \xrightarrow[\text{open}]{} S$.

then form the diagram:

$$\begin{tikzcd}
S''' = S' \times S' \times S' \arrow[d, shift left=2, "p_{13}"] \arrow[d, "p_{23}"] \arrow[d, shift right=2, "p_{12}"'] \\
S'' = S' \times_S S' \arrow[d, shift left, "p_1"] \arrow[d, shift right, "p_2"'] \\
S' \arrow[d, "p"] \\
S
\end{tikzcd}
\qquad
\text{e.g.}\quad
\begin{tikzcd}
(a,b,c) \arrow[d, "p_{23}"] \\
(b,c) \arrow[d, "p_2"] \\
c
\end{tikzcd}$$

__Def__: A __glueing datum__ for an object $x \in F(S')$ is an isomorphism in $F(S') \underset{p_2^*}{\overset{p_1^*}{\rightrightarrows}} F(S'')$ $p_1^* x \xrightarrow[\sim]{\varphi} p_2^* x$. A __descent datum__ is a glueing datum (x, φ) which satisfies the cocycle condition $p_{13}^* \varphi = pr_{23}^* \varphi \circ pr_{12}^* \varphi$. The category of descent data of F w.r.t. $S' \to S$, $F(S' \to S) = \{ (x, \varphi) \mid (x,\varphi) \text{ a descent datum} \}$ where maps $(x,\varphi) \to (x', \varphi')$ are $x \xrightarrow{f} x'$ s.t.

$$\begin{tikzcd}
p_1^* x \arrow[r, "\varphi"] \arrow[d, "p_1^* f"'] & p_2^* x \arrow[d, "p_2^* f"] \\
p_1 x' \arrow[r, "\varphi'"] & p_2^* x'
\end{tikzcd}$$

commutes.

__FACT__: There is a natural morphism $F(S) \to F(S' \to S)$.
$x \longmapsto (p^* x, \text{can})$

$S'' \underset{p_2}{\overset{p_1}{\rightrightarrows}} S' \xrightarrow[p]{} S$

since $p \circ p_1 = p \circ p_2$ so $x \mapsto p_1^* p^* x \xrightarrow[\sim]{\text{can}} p_2^* p^* x$

Why is $(p^* x, \text{can})$ a descent datum?

$$\begin{array}{c} S''' = S' \times_S S' \times_S S' \\ \downarrow p_{12} \quad \downarrow p_{13} \quad \downarrow p_{23} \\ S'' = S' \times_S S' \\ p_1 \downarrow \qquad \downarrow p_2 \\ S' \\ \downarrow p \\ S \end{array}$$

$p_1 \circ p_{12} = p_1 \circ p_{13}$ $\qquad$ $p_{13}^* \mathrm{can}$

$$\begin{tikzcd} p_{12}^* p_1^* p^* \mathrm{can} \arrow[r, equal] \arrow[d, "p_{12}^* \mathrm{can}"] & p_{13}^* p_1^* p^* X \arrow[r] & p_{13}^* p_2^* p^* X \arrow[d, equal, "p_2 \circ p_{13} = p_2 \circ p_{23}"] \\ p_{12}^* p_2^* p^* X \arrow[r, equal] & p_{23}^* p_1^* p^* X \arrow[r, "p_{23}^* \mathrm{can}"] & p_{23}^* p_2^* p^* X \end{tikzcd}$$

$p_1 \circ p_{23} = p_2 \circ p_{12}$

Ex: Consider $\mathrm{BGL}_n : (\mathrm{Sch}_S)_{Zar} \to \mathrm{Grp}$, the category of descent data w.r.t. a open covering $\{U_i \to S\}$ ($\Leftrightarrow \bigsqcup U_i \to S$)

$\mathrm{BGL}_n(\sqcup U_i \to S)$ is $\{(V_i, \varphi_{ij}) \mid V_i$ v.b. of rk n on U_i, $\varphi_{ij} : V_i|_{U_i \cap U_j} \xrightarrow{\sim} V_j|_{U_i \cap U_j}\}$ s.t. on $U_i \cap U_j \cap U_k$ we have $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$.

Now we can define what a stack is

Def: Let $F : (\mathrm{Sch}_S)_* \to \mathrm{Grp}$ be a functor on a site $* \in \{$zar, etale, fppf$\}$

F is a stack if $\boxed{1}$ $\forall T \in \mathrm{Sch}_S$ & all $a, b \in F(T)$ the functor $\underline{\mathrm{Hom}}(x, y) : (\mathrm{Sch}/T)_* \to \mathrm{Set}$ is a sheaf.

$\boxed{2}$ For every $T \in \mathrm{Sch}_S$ & every cover $T' \to T$ every descent datum $(x, \varphi) \in F(T' \to T)$ is effective (i.e. in the essential image of $F(T) \to F(T' \to T)$)

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1) + 2) $\Leftrightarrow$ $F(T) \to F(T' \to T)$ is an equivalence $\forall T \in \mathrm{Sch}_S$ & all covers $T' \to T$

Does this fix the problem for BGL_n? $\mathcal{M}_g$?

Just like for manifolds it is not easy to show a moduli problem is a stack.

Two important examples

$\boxed{\text{Thm}}$ (Groth.) $\underline{QCoh} : (\underline{Sch}_S)_{fppf} \to \underline{Grp}$, $T \mapsto QCoh(T)$ is a stack.

$\boxed{\text{Cor}}$ $BGL_n = \text{Vect}_n : (\underline{Sch})_S \to \underline{Grp}$ sending $(T \to S) \longmapsto \{V \mid V \text{ vect on } T$

is a stack.

Pf: Observe

$$\begin{tikzcd}
BGL_n(T) \arrow[r, hook, "\text{fully}", "\text{faithful}"'] \arrow[d] & QCoh(T) \arrow[d, "2"] \\
BGL_n(T' \to T) \arrow[r, hook, "\text{fully}", "\text{faithful}"'] & QCoh(T' \to T)
\end{tikzcd}$$

By the example above.

So left vertical arrow is fully faithful. It remains to show that if M is an R-module which after a faithfully flat base change $R \to R'$ $M \otimes R'$ is locally free of rk n, then M is locally free of rank n.

An important stack in algebraic geometry.

Def: $\underline{Pol} : \underline{Sch}_S \to \underline{Grp}$ is the functor which sends $T \longmapsto$

$$\left\{ \begin{array}{c} (X, L) \\ \downarrow f \\ T \end{array} \;\middle|\; \begin{array}{l} \bullet\ f \text{ proper, flat, finite presentation} \\ \bullet\ L \text{ rel. ample w.r.t. } f \end{array} \right\}$$

Morphisms:

$$\begin{array}{c} (X', L') \\ \uparrow (a,b) \\ (X, L) \end{array} \qquad \begin{tikzcd}[column sep=small] X \arrow[rr, "a"] \arrow[dr, "f"'] & & X' \arrow[dl, "f'"] \\ & T & \end{tikzcd} \qquad b : a^*L' \xrightarrow{\sim} L.$$

$\boxed{\text{Cor}}$ $\underline{Pol}$ is a stack.

Cor: $\mathcal{M}_g$ is a stack for $g \neq 1$.

Fix a cover $S' \xrightarrow{p} S$

Pf: Want to show that $\mathcal{M}_g(S) \to \mathcal{M}_g(S' \to S)$ is an equivalence

Step #1: Show it is fully faithful i.e. $\mathrm{Hom}(C, D) \xrightarrow[\sim]{F} \mathrm{Hom}((C', can), (D', can))$

$f \mapsto \varphi$

$$\begin{tikzcd} p_1^* p^* C \arrow[r, "can"] \arrow[d, "p_1^* f"'] \arrow[dr, phantom, "\circlearrowleft"] & p_2^* p^* C \arrow[d, "p_2^* f"] \\ p_1^* p^* D \arrow[r, "can"] & p_2^* p^* D \end{tikzcd}$$

Recall that $h_D : \underline{Sch_S} \to \underline{Set}$ is a sheaf

$\Rightarrow h_D(C) \to h_D(C') \rightrightarrows h_D(C'')$ is exact

$$\begin{matrix} \| & & \| \\ \mathrm{Hom}_S(C, D) & \to & \mathrm{Hom}_{S'}(C', D') & \underset{pr_2}{\overset{pr_1}{\rightrightarrows}} & \mathrm{Hom}_{S''}(C'', D'') . \end{matrix}$$

$$\begin{tikzcd} pr_1 p^* C \arrow[r, "pr_1^* f"] \arrow[d, equal, "can"'] \arrow[dr, phantom, "\circlearrowleft"] & pr_1^* p^* D \arrow[d, equal, "can"] \\ pr_2 p^* C \arrow[r, "pr_2^* f"] & pr_2^* p^* D \end{tikzcd}$$

F is injective b/c
$\mathrm{Hom}((C', can), (D', can)) \subseteq \mathrm{Hom}(C', D')$

& surjective b/c the kernel is exactly $\mathrm{Hom}((C', can), (D', can))$. So it is fully faithful.

To show $\mathcal{M}_g(S) \to \mathcal{M}_g(S' \to S)$ is essentially surjective, need to show every descent datum is effective. Here is where we use $g \neq 1$ b/c $\omega_{C/S}^\vee$ or $\omega_{C/S}$ is rel. ample (if $g = 0$ or if $g \geq 2$). This means we have

$$\mathcal{M}_g \xrightarrow{g \geq 2} \mathcal{P}ol$$

$$\begin{matrix} C \\ \downarrow \\ S \end{matrix} \longmapsto \left(\begin{matrix} C \\ \downarrow \\ S \end{matrix} , \omega_{C/S} \right)$$

b/c there is a nat'l pullback map on forms.

Moreover

$$\begin{tikzcd} \mathcal{M}_g(S) \arrow[r] \arrow[d] & \mathcal{P}ol(S) \arrow[d, "2"] \\ \mathcal{M}_g(S' \to S) \arrow[r] & \mathcal{P}ol(S' \to S) \end{tikzcd}$$

So given a descent datum $\left(\begin{matrix} C' \\ \downarrow \\ S' \end{matrix}, \varphi \right) \mapsto ((C', \omega_{C'/S'}), \varphi)$ which goes via to a family $\left(\begin{array}{c} C \\ \downarrow \\ S \end{array}, L\right)$ s.t. $\begin{array}{c} C' \cong C \times_S S' \\ \downarrow \\ S' \to S \end{array}$ $\Rightarrow$ $C \to S$ is flat proper finite presentation w/ smooth curves of genus g as fibers. So

$M_g(S) \to M_g(S' \to S)$ is essentially surjective. $\blacksquare$

<u>Thm</u> (Raynaud) M_1 is not a stack!!!
