Ejercicios Stacks

Exercise 4 (First Set):

Descent for Quasi-coherent sheaves.

The purpose of the following sequence of exercises is to prove the following:

Theorem (Grothendieck):

Let $p : S' \to S$\$ be a faithfully flat and quasi-compact morphism of schemes. Then the pullback functor $\mathcal{F} \mapsto p^*\mathcal{F}$ which sends a quasi-coherent $S{-}moduletoaquasi - coherent\$S'$-module with descent data:

$$p^* : Qcoh(S) \to Qcoh(S' \to S)$$

is an equivalence of categories.

This is foundational and essentially all descent-related results (e.g. Galois descent) and more can be derived from this Theorem.

Notation:

For a *descent datum* of $Qcoh(S' \to S)$ we will use $\mathcal{F}' \in Qcoh(S')$ along with an isomorphism $\varphi : p_1^*(\mathcal{F}') \to p_2^*(\mathcal{F}')$ of quasi-coherent sheaves over $S'' = S' \times_S S'$ where $p_i : S' {\to} S'$ for $i = 1, 2$ are the projections. The descent datum also satisfies the *cocycle condition*

$$p_{13}^*(\varphi) = p_{23}^*(\varphi) \circ p_{12}^*(\varphi)$$

where $p_{i,j} : S''' \to S''$ for $1 \leq i < j \leq 3$ are the projections onto two coordinates from $S''' = S' \times_S S' \times_S S'$.
A *morphism of descent datums* is a morphism of quasi-coherent sheaves $f : \mathcal{F}' \to \mathcal{G}'$ over S' with isomorphisms φ and φ' resp. such that the following diagram commutes:

$$\begin{tikzcd} p_1^*(\mathcal{F}') \arrow[r, "\varphi", "\sim"'] \arrow[d, "p_1^*(f)"'] & p_2^*(\mathcal{F}') \arrow[d, "p_2^*(f)"] \\ p_1^*(\mathcal{G}') \arrow[r, "\sim", "\varphi'"'] & p_2^*(\mathcal{G}') \end{tikzcd}$$

Proof:

Part a): Fully faithfulness:

We start by noting that for any quasi-coherent sheaf $\mathcal{F}$ over S, we have that $p_1^*(p^*(\mathcal{F})) = p_2^*(p^*(\mathcal{F}))$ as $p_1 \circ p = p_2 \circ p$. This implies that if we start with a morphism $f : \mathcal{F} \to \mathcal{G}$ of quasi-coherent sheaves over S, then $p_1^*(p^*(f)) = p_2^*(p^*(f))$ as well.
So, to achieve fully faithfulness, all morphisms of canonical descent datums coming from quasi-coherent sheaves over S must be morphisms of quasi-coherent sheaves over S' that are the same morphisms when taking pull-backs over the projections of S''.
This can be summarized in the following exact sequence:

$$0 \to \mathrm{Hom}_S(\mathcal{F}, \mathcal{G}) \xrightarrow{p^*} \mathrm{Hom}_{S'}(p^*\mathcal{F}, p^*\mathcal{G}) \xrightarrow{p_1^* - p_2^*} \mathrm{Hom}_{S''}(q^*\mathcal{F}, q^*\mathcal{G})$$

where $q = p_1 \circ p = p_2 \circ p$.

Part 1: Making the image affine:

The set $\mathrm{Hom}(\mathcal{F}, \mathcal{G})$ corresponds to $\Gamma(\mathcal{H}om(\mathcal{F}, \mathcal{G}), S)$, i.e., the global sections of the Hom sheaf. As it is a sheaf over the Zarisky topology on S, if the exact sequence is true for any open subset $U \in S$, then the statement is true over the whole of S by the sheaf property of this sheaf and the fact that taking global sections corresponds to a left exact functor. Thus, we can assume $S = \mathrm{Spec}(R)$ for a commutative ring R.

Part 2: Making the domain affine:

Now, as p is quasi-compact, the preimage on any quasi-compact subset of S is quasi-compact. As S is affine, it is quasi-compact, thus so is S' allowing us to cover it by finitely many affine open subschemes $S' = \bigcup_i S'_i$.
So let $\bar{S}'$ be the disjoint union of the S_i with $u : \bar{S}' \to S'$ being the canonical morphism, it is quasi-compact and faithfully flat as any open immersion $S'_i \to S'$ is flat. We will also consider $\bar{p} = f \circ p : \bar{S}' \to S$ and $\bar{S}'' = \bar{S}' \times_S \bar{S}'$ with projections $\bar{p}_i$ for $i = 1, 2$ and note that $\bar{p}$ is faithfully flat. and quasi-compact
In summary, we have the following diagram:

$$\begin{tikzcd}
\bar{S}'' \arrow[r, shift left, "\bar{p}_1"] \arrow[r, shift right, "\bar{p}_2"'] \arrow[d, "v = u \times u"'] & \bar{S}' \arrow[d, "u"'] \arrow[dr, "\bar{p}"] & \\
S'' \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & S' \arrow[r, "p"'] & S
\end{tikzcd}$$

As fpqc morphisms for a Grothendieck Topology, it is not hard to see that v is fpqc as well by composing coverings.
This translates to the following diagram at the level of morphisms of quasi-coherent sheaves:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_S(\mathcal{F}, \mathcal{G}) \arrow[r, "p^*"] \arrow[d, equal] & \mathrm{Hom}_{S'}(p^*\mathcal{F}, p^*\mathcal{G}) \arrow[r, "p_1^* - p_2^*"] \arrow[d, hook, "u^*"] & \mathrm{Hom}_{S''}(q^*\mathcal{F}, q^*\mathcal{G}) \arrow[d, hook, "v^*"] \\
0 \arrow[r] & \mathrm{Hom}_S(\mathcal{F}, \mathcal{G}) \arrow[r, "\bar{p}^*"] & \mathrm{Hom}_{\bar{S}'}(\bar{p}^*\mathcal{F}, \bar{p}^*\mathcal{G}) \arrow[r, "\bar{p}_1^* - \bar{p}_2^*"'] & \mathrm{Hom}_{\bar{S}''}(\bar{q}^*\mathcal{F}, \bar{q}^*\mathcal{G})
\end{tikzcd}$$

where all vertical arrows are injective as the pull-back functors of quasi-coherent sheaves induced by u and v are faithful. This diagram easily implies that if the lower sequence is exact, the upper one is as well, as any morphism of sheaves in the lower Hom sets that is equal to 0 must come from a 0 morphism on the corresponding upper Hom set as long as they belonged to the image of u^* or v^*.
In conclusion, as $\bar{S}'$ is affine, we can assume $S' = \mathrm{Spec}(R')$ is affine.

Part 3: Reducing to a problem for modules:

As both S and S' are affine, quasi-coherent modules become modules over R and R'. If M and N correspond to $\mathcal{F}$ and $\mathcal{G}$ respectively, then the exact sequence we need to prove is:

$$0 \to \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_{R'}(M \otimes_R R', N \otimes_R R') \stackrel{p_1^*-p_2^*}{\to} \mathrm{Hom}_{R''}(M \otimes_R R'', N \otimes_R R'')$$

where $R'' = R' \otimes_R R'$.
But the second and third terms can be rewritten as:

$$0 \to \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(M, N \otimes_R R') \stackrel{p_1^*-p_2^*}{\to} \mathrm{Hom}_R(M, N \otimes_R R'')$$

where we are considering $N \otimes_R R'$ and $N \otimes_R R''$ as R-modules using the morphisms from R to R' and R'' respectively. But this sequence is exact if the following sequence is exact:

$$0 \to N \to N \otimes_R R' \to N \otimes_R R''$$

as the functor $\mathrm{Hom}_R(M, \cdot)$ is left exact. This sequence in turn, comes from the sequence

$$0 \to R \to R' \stackrel{p_1^*-p_2^*}{\to} R''$$

where the third arrow corresponds to the morphism $r' \mapsto r' \otimes 1 - 1 \otimes r'$. So, to finish, we need to show that for any R-module N the sequence

$$0 \to N \to N \otimes_R R' \to N \otimes_R R''$$

arising from tensoring with N over the sequence with R, R' and R'' is exact.

Part 4: End of the proof:

Let us start with the sequence:

$$0 \to R \stackrel{p}{\to} R' \stackrel{p_1^*-p_2^*}{\to} R''$$

if we tensor by R', as it is flat over R we have the following sequence:

$$0 \to R' \to R'' \to R'''$$

where R''' is the tensor product of three copies of R' over R.
In this case, we naturally have a surjective morphism $R'' \to R'$ corresponding to a section $S' \to S''$. As R' is faithfully flat over R, if this second sequence with a section is exact, our initial sequence is exact as well so we can assume we have a section $s : R' \to R$ on the first sequence, the sections satisfies $id = s \circ p$.
Let $r' \in R'$ be an element in the kernel of $p_1^* - p_2^*$, this means that $r' \otimes 1 = 1 \otimes r'$ in R''. If we apply $s \otimes id_{R'} : R'' \to R \otimes_R R'$ to this equation we obtain

$$s(r') \otimes 1 = s(1) \otimes r'$$

but $R \otimes_R R'$ is canonically isomorphic to R' via $p \otimes id_{R'}(a \otimes b) = p(a)b$, thus, if we call $r = s(r') \in R$, we have

$$p(r) = r'$$

implying that r' belonged to R which shows that the sequence is exact.
Now, if we take any R-module N, we can tensor the exact sequence we just obtain to get the sequence:

$$N \to N \otimes_R R' \overset{id_N \otimes (p_1^* - p_2^*)}{\to} N \otimes_R R''$$

as N is not necessarily flat, we cannot even state that the first arrow is injective yet. Let us suppose we have a section $s : R' \to R$ using the fact that R' is faithfully flat over R, in this case the composition

$$N \cong N \otimes_R R \overset{id_N \otimes p}{\to} N \otimes_R R' \overset{id_N \otimes s}{\to} N$$

is the identity id_N, in particular the first arrow in the former sequence is injective. The other part of the exact sequence follows a similar argument using $id_N \otimes s \otimes id_{R'}$ over an element $\sum n_i \otimes r'_i \in N \otimes_R R'$ belonging to the kernel of $id_N \otimes (p_1^* - p_2^*)$ that shows that it came from N via $id_N \otimes p$, finishing the proof of fully faithfulness.

Part b): Essential subjectivity:

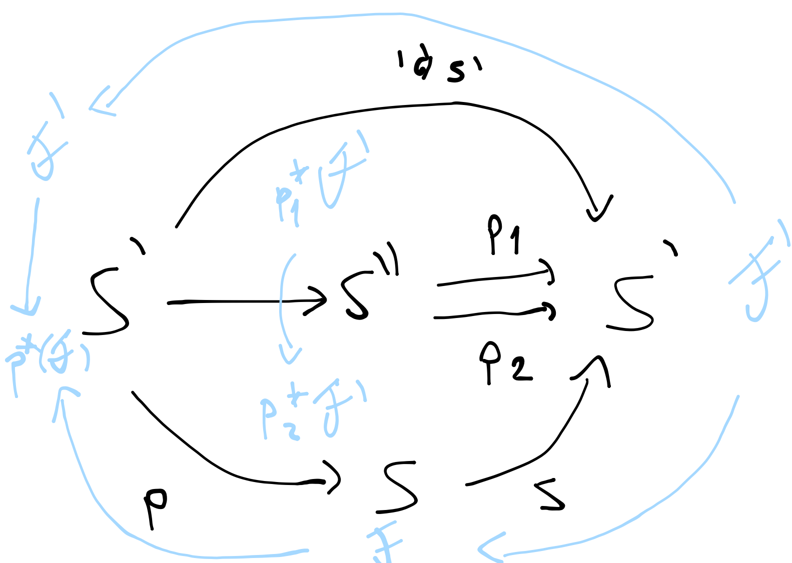
Now given a descent datum $\mathcal{F}'$ in $Qcoh(S' \to S)$, we will show that it is isomorphic to a descent datum of the form $p^*(\mathcal{F})$ for some $\mathcal{F} \in Qcoh(S)$.

Part 1: The case with a section:

Let us assume we have a section $s : S \to S'$ with $p \circ s = id_S$, in this case we will show that any descent datum is effective: Let $\varphi : p_1^*(\mathcal{F}') \to p_2^*(\mathcal{F}')$ be the isomorphism on S'', in this case we have a commutative diagram

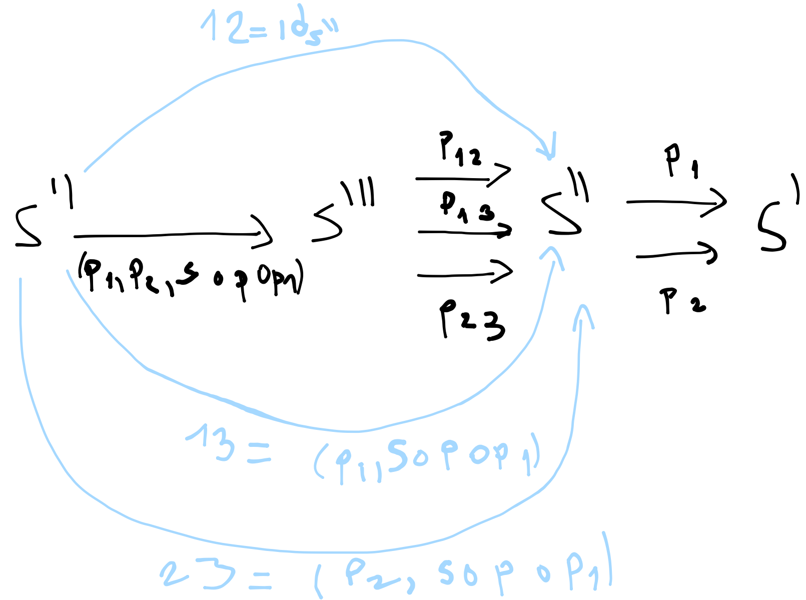
$$\begin{tikzcd} & S' \arrow[dr, "p"] & \\ S' \arrow[ur, "s\circ p"] \arrow[dr, "id_{S'}"'] & & S \\ & S' \arrow[ur, "p"'] & \end{tikzcd}$$

so we can define the morphism $(id_{S'}, s\circ p) : S' \to S''$ and pull-back the ismorphism φ along it to obtain an isomorphism of quasi-coherent sheaves $f : \mathcal{F}' \to p^*(\mathcal{F})$ where $\mathcal{F} = s^*(\mathcal{F}')$ using this handy commutative diagram



To finish, we need to show that f is a morphism of descent datums, so it must satisfy the following equality $p_1^*(f) = p_2^*(f)\circ\varphi$. To show this, we will use the cocycle condition of φ, which states that $p_{13}^*(\varphi) = p_{23}^*(\varphi)\circ p_{12}^*(\varphi)$ where $p_{i,j} : S''' \to S''$ are the canonical projections onto two coordinates.
The equality required to have a morphism of descent datums comes from using the morphism $(p_1, p_2, s\circ p\circ p_1) : S'' \to S'''$. We claim that after pulling back the cocycle

condition over this morphism, we will obtain $p_1^*(f) = p_2^*(f) \circ \varphi$: Indeed, by looking at the following diagram



Thus, to pull-back the cocycle condition for φ all the way to the leftmost copy of S'' we need to pull back $\mathcal{F}'$ along the three blue compositions which determine the domain and range of the pull-backs for φ and use that $f = (id_{S'}, s \circ p)^*(\varphi)$ to obtain the resultant isomorphism over S''. Thus, we have

$$\begin{array}{rrl} 12 \to & \varphi: & p_1^*(\mathcal{F}') \to p_2^*(\mathcal{F}') \\ 13 \to & p_1^*(f): & p_1^*(\mathcal{F}') \to p_1^*(\mathcal{F}) \\ 23 \to & p_2^*(f): & p_2^*(\mathcal{F}') \to p_1^*(\mathcal{F}) = p_2^*(\mathcal{F}) \end{array}$$

and the cocycle condition for φ

$$p_{13}^*(\varphi) = p_{23}^*(\varphi) \circ p_{12}^*(\varphi)$$

becomes

$$p_1^*(f) = p_2^*(f) \circ \varphi$$

after pulling back, showing that we actually have a morphism of descent datums, showing that we have effective descent in this case.

Part 2: Reducing the general case to a problem of modules:

Now we do not assume that we have a section for $p : S' \to S$ and we would like to show that can reduce ourselves to the case when S and S' are affine, so let us assume that we have effective descent for any fpqc cover between affine schemes.
If $\mathcal{F}'$ is a descent datum for p, let us suppose that for any affine open subscheme $U \subset S$, we have effective descent for the covering $\bar{p} : S' \otimes_S U = U' \to U$, meaning that there is a sheaf $\mathcal{F}_U$ over U with an isomorphism of descent datums $\mathcal{F}'|_{U'} \cong \bar{p}^*(\mathcal{F}_U)$. Now, if we take

an affine cover $\{U_i\}$ of S, we can arrange the sheaves $\mathcal{F}_{U_i}$ such that they are isomorphic over the double intersections and satisfy the cocycle condition for any triple intersection of covers, using the fact that $\mathcal{F}'|_{U_i'} \cong p_i^*(\mathcal{F}_{U_i})$ for all i. Thus, we can glue these sheaves (Hartshorne Ch. II Exercise 1.22) to obtain a sheaf $\mathcal{F}$ over S with an isomorphism $f : \mathcal{F}' \to p^*(\mathcal{F})$ of descent data coming from the fully faithfulness of the canonical functor over the fpqc cover $\coprod V_i \to S'$ where $V_i = p^*(U_i)$ which are finite unions of affine schemes as p is quasi-compact.
So we conclude that it suffices to show the effective descent property over affine schemes.

Part 3: Effective descent of modules (part A):

Now let us suppose $S = \mathrm{Spec}(R)$ and $S' = \mathrm{Spec}(R')$. A descent datum over S' is a R'-module M' together with an isomorphism $\varphi : M' \otimes_R R' \to R' \otimes_R M'$ (here we have one less prime as the isomorphism over R'' holds if and only if it does with one less copy of R') that satisfies the cocycle condition, that we can consider over R''.
Now, let us consider the canonical morphism $a : M' \to M' \otimes_R R'$ defined as $m \mapsto m \otimes 1$ and let $b : M' \to M' \otimes_R R'$ be $m \mapsto 1 \otimes m \in R' \otimes_R M' \mapsto \varphi^{-1}(1 \otimes m) \in M' \otimes_R R'$.
To finish the proof we are going to prove the following statement:
M' descends effectively to a R-module if and only if there exists a R-module K that fits into the following exact sequence:

$$0 \to K \to M' \overset{a-b}{\to} M' \otimes_R R'$$

and the natural map $K \otimes_R R' \to M'$ is an isomorphism.
The only if $(\Leftarrow)$ statement is clearly true, so let us suppose that M' descends effectively to a R-module M with $M \otimes_R R' \cong M'$. Now let $m' \in M$ be an element of M' that we can write as $m' = \sum_i m_i \otimes r'$, in this case the isomorphism φ is just $m \otimes r_1' \otimes r_2' \mapsto r_2' \otimes m \otimes r_1'$ and we can readily check that $a(m') = b(m')$ making the sequence

$$0 \to M \to M' \overset{a-b}{\to} M' \otimes_R R'$$

exact.

Part 3: Effective descent of modules (part B):

To finish, we need to find the module K, let $K' = ker(a-b)$ viewed as a R'-module and let K be the restriction to R of K'. In this case we clearly have an exact sequence

$$0 \to K \to M' \overset{a-b}{\to} M' \otimes_R R'$$

so we just need to show that $K \otimes_R R' \cong M'$ as modules over R'.
In the sequence above we are seeing all elements as R-modules, and if we tensor by R'

which is faithfully flat over R, we obtain an exact sequence of R' modules:

$$0 \to K \otimes_R R' \to M' \otimes_R R' \overset{(a-b)\otimes id_{R'}}{\to} M' \otimes_R R''$$

but in this case, the isomorphism $\varphi : M' \otimes_R R' \to R' \otimes_R M'$ becomes $t \circ (\phi \otimes id_{R'}) : M' \otimes_R R'' \to R'' \otimes_R M'$ where $t : R' \otimes_R M' \otimes_R R' \to R'' \otimes_R M'$ is the canonical isomorphism defined as $t(r_1' \otimes m' \otimes r_1') = r_1' \otimes r_2' \otimes m'$ giving us a descent datum for the morphism $R' \to R''$, but this morphism has a section, so there exists an effective descent $\bar{M}$ over R' with $\bar{M} \otimes_{R'} R'' \cong M' \otimes_R R''$ which also implies that $\bar{M} \cong M' \otimes_R R'$. By the last part, $\bar{M}$ also fits into an exact sequence

$$0 \to \bar{M} \to M' \otimes_R R' \overset{(a-b)\otimes id_{R'}}{\to} M' \otimes_R R''$$

and moreover, we have the following diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & K \otimes_R R' \arrow[r] \arrow[d] & M' \otimes_R R' \arrow[r] \arrow[d] & M' \otimes_R R'' \arrow[d] \\
0 \arrow[r] & \bar{M} \arrow[r] & M' \otimes_R R' \arrow[r] & M' \otimes_R R''
\end{tikzcd}$$

but the second and third vertical arrows are isomorphisms, which shows that $K \otimes_R R' \cong \bar{M}$ as modules over R' by chasing the diagram, this concludes the proof as we have $K \otimes_R R' \cong \bar{M} \cong M' \otimes_R R'$ showing effective descent.