Algebraic stacks & Examples

Lecture #3

We have outlined the <u>topological</u> properties that we would like a moduli problem $F: \underline{Sch}^{op}_{fppf} \to \underline{Grp}$ to have [objects & morphisms glue), but we want to do geometry, so they must <u>locally</u> have an algebraic form.

Fix a base scheme $S = \operatorname{Spec}\mathbb{Z}$.

<u>Def</u>: A stack $F: \underline{Sch}_{S_{fppf}} \to Grp$ is algebraic if

(i) $F \xrightarrow{\Delta} F \times_S F$, $a \longmapsto (a, a, id)$ is representable by schemes.

[$\iff$ $\forall T \in Sch_S$ & $\forall a, b \in F(T)$ the functor of isomorphisms $\underline{Isom_T(a,b)}: \underline{Sch}_{T_{fppf}} \to \underline{Set}$, $[T' \to T] \longmapsto Isom(a|_{T'}, b|_{T'})$ is representable]

"The distance between F & a scheme" is algebraic

(ii) $\exists$ a smooth surjective $U \to F$ where U is scheme.

[a map $U \to F$ can be checked to be sm./surj using (i)! i.e. it is sm./surj.
iff $\forall T \to F$ with T sch,

$$\begin{tikzcd} U_T \arrow[r] \arrow[d, "sm./surj."'] & U \arrow[d] \\ T \arrow[r] & F \end{tikzcd}$$
]

F is $\boxed{\text{Deligne-Mumford}}$ if $\exists U \to F$ étale + surj. $\boxed{\text{Sep'd}}$ if Δ is proper

---

$\boxed{\text{Ex \#1}}$ The stack $BGL_{n,S} = Vect_{n,S} : \underline{Sch}_{S_{fppf}} \to \underline{Grp}$, $T \longmapsto$ grp. of rk n v.b.'s on T, is algebraic:

(i) Given $V, W \in Vect_n(T)$, $\underline{Isom}_T(V, W) \longrightarrow T$ is Zar. locally on T $\cong$ $\underline{Isom}_{T'}(\mathcal{O}_{T'}^{\oplus n}, \mathcal{O}_{T'}^{\oplus n}) = \underbrace{GL_{n,T'} \longrightarrow T'}_{\text{affine scheme}}$ $\Longrightarrow$ Δ_{BGL_n} rep'ble.

(ii) $S \xrightarrow{\mathcal{O}^{\oplus n}} BGL_{n,S}$ is smooth & surjective

<u>Pf</u>: Indeed, if

$$\begin{tikzcd} \underline{Isom}_T(\mathcal{O}^{\oplus n}, T) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & S \arrow[d, "triv."] \\ T \arrow[r, "V"'] & BGL_{n,S} \end{tikzcd}$$

& $\underline{Isom}_T(\mathcal{O}^{\oplus n}, T) \to T$ is Zar locally $\cong GL_{n,T}$.

$\Longrightarrow$ $S \to BGL_{n,S}$ is smooth + surj.

② $\mathcal{M}_g$ is an algebraic stack for $g \geq 2$, in fact Deligne-Mumford + sep'd

(GIT-esque)
Idea: There is locally closed locus $H_g \subseteq \mathrm{Hilb}_{\mathbb{P}^{5g-6}}$ whose pts. are exactly those closed subschemes $Y \hookrightarrow \mathbb{P}^{5g-6}$ s.t.

i) Y is a curve of genus g (smooth + geom. connected)

ii) $Y \hookrightarrow \mathbb{P}^{5g-6}$ is tri-canonically embedded.

Then $\exists$ map $H_g \xrightarrow{\pi} \mathcal{M}_g$ sending $Y \hookrightarrow \mathbb{P}^{5g-6} \mapsto Y$, also

1) π is surjective on $\bar{k}$-pts.

2) If
$$\begin{tikzcd} H_g \times_{\mathcal{M}_g} R \arrow[r] \arrow[d] & H_g \arrow[d] \\ \underbrace{\mathrm{Spec}\, R}_{\text{complete local ring}} \arrow[r] & \mathcal{M}_g \end{tikzcd}$$
then $H_g \times_{\mathcal{M}_g} R \cong PGL_{5g-6} \times R$

$\Rightarrow \pi$ is smooth.

Another more intrinsic "local" proof exists.

Proofs of this will be outlined in the exercises (Maybe) ▨

Next: the arguably most important class of algebraic stacks.

Quotient stacks Note that $H_g \to \mathcal{M}_g$ is PGL_{5g-6}-invariant & has PGL_{5g-6} as fibers. So this suggests $\mathcal{M}_g$ is the quotient of H_g by PGL_{5g-6}.

We need:

Def: Let G be a sheaf of grps (Sfppf), T a sheaf (Sfppf), it is a (left) G-torsor if

(i) $\exists$ action $G \times T \to T$. (ii) $\exists$ cover $S' \to S$ s.t. $T|_{S'} \xrightarrow{\sim} G|_{S'}$

Ex: - $G \times S \to S$ (trivial G-torsor)

- $\mathrm{Spec}\, L \to \mathrm{Spec}\, k$ where $[L:K]$ is Galois is a $\mathrm{Gal}(L/k)$-torsor

- $C \to S$ genus 1 curve torsor under Jacobian.

- If $V \to S$ is a vect. bundle (rk n) then $\underline{\mathrm{Isom}}(\mathcal{O}^{\oplus n}, V) \to S$ is a GL_n-torsor.

Claim: $T \xrightarrow[G\text{-tor}]{} S$ trivial $\Leftrightarrow \exists$ section. Pf: ($\Leftarrow$) section defines a G-equivariant map $G \to T$, iso. b/c it's locally. ($G \to T$, $\searrow S \nearrow x$)

Quotient stacks are useful b/c they remember automorphisms & their algebraicity is easy

Def: Let $G/_k$ be a smooth + affine group scheme acting on X scheme

then the associated quotient stack (*) is denoted $[X/G]$

and its S pts are diagrams
$$\begin{tikzcd} T \arrow[r, "G\text{-equivariant}", "\varphi"'] \arrow[d, "G\text{-torsor}"] & X \\ S \end{tikzcd}$$
+ a

G-equivariant S-map
$$\begin{tikzcd} T' \arrow[drr, "\varphi'"] \arrow[ddd, bend right] & & \\ & & \\ T \arrow[uu, dotted, "\phi"] \arrow[rr, "\varphi"'] \arrow[d] & & X \\ S \end{tikzcd}$$
making the diagram commute.

Lets unravel this if $S = \operatorname{Spec} \mathbb{C}$ (or any alg. closed field), what is the category $[X/G](\mathbb{C})$

objects?
$$\begin{tikzcd} T \arrow[r, "G\text{-equivariant}"] \arrow[d, "G\text{-torsor}"] & X \\ \operatorname{Spec}\mathbb{C} \end{tikzcd}$$
All G-torsors trivial / alg. closed field

so $T \cong G$. (choice equiv to choosing a section)

$$\begin{tikzcd} G \arrow[r, "\text{equivariant}", "p"'] \arrow[d] & X \\ \operatorname{Spec}\mathbb{C} \end{tikzcd}$$
(a G-equiv. map $G \to X$ is the same as a map $\operatorname{Spec}\mathbb{C} \xrightarrow{p} X$) given $p: \operatorname{Spec}\mathbb{C} \to X$

$G \to X$

$g \mapsto g \cdot p$.

Morphisms:
$$\begin{tikzcd} G \arrow[dr, "q"] & \\ G \arrow[u, "\cdot g"] \arrow[r, "p"'] & X \end{tikzcd} \Rightarrow \begin{tikzcd} 1 \arrow[dr, mapsto] \arrow[d, mapsto] & q \\ g \arrow[r, mapsto] & g\cdot p \arrow[u, equal] \end{tikzcd} \longleftrightarrow \{ g \in G \mid g \cdot p = q \}.$$

in particular $\operatorname{Aut}\left(\begin{tikzcd} G \arrow[r, "p"] \arrow[d] & X \\ S \end{tikzcd} \right) = \operatorname{Stab}(p)$ & $|[X/G](\mathbb{C})| =$ orbits of $X(\mathbb{C})$ under $G(\mathbb{C})$.

Thm: If $G \to \operatorname{Spec} k$ smooth gp. scheme, $G \curvearrowright X$ $\Rightarrow$ $[X/G]$ is an algebraic stack.

Pf: (i) $\underline{\mathrm{Isom}}\left(\begin{smallmatrix} T \to X \\ \downarrow \\ S\end{smallmatrix}, \begin{smallmatrix} T' \to X \\ \downarrow \\ S\end{smallmatrix}\right)$ is locally $I = \underline{\mathrm{Isom}}\left(\begin{smallmatrix} G \xrightarrow{p} X \\ \downarrow \\ S\end{smallmatrix}, \begin{smallmatrix} G \xrightarrow{q} X \\ \downarrow \\ S\end{smallmatrix}\right)$ where $p, q \in X(S)$.
$\downarrow$
S

$$\begin{tikzcd} I \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d, "\text{loc. closed}"] \\ G \arrow[r, "(c_p, q)"'] & X \times X \end{tikzcd}$$

this is the set $\{g \in G \mid g \cdot p = q\} \subseteq G$ $\qquad g \longmapsto (p, g \cdot q)$

So rep'ble b/c Δ is locally closed. [Using fppf descent for quasi-affine morphisms]

(ii) $X \xrightarrow{\pi} [X/G]$ defined by $\begin{matrix} G \times X \xrightarrow{m} X \\ \downarrow \\ X \end{matrix}$ is a G-torsor!

Indeed

$$\begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d] \\ T \arrow[r] & {[X/G]} \end{tikzcd}$$

$\begin{matrix} P \to X \\ \downarrow \\ T \end{matrix}$

(check this!) Exercise.

$\Rightarrow$ π is smooth.

So these are stack quotients, how do they relate to scheme-quotients when they exist?

Big problem in AG b/c this is how moduli spaces are constructed.

In many situations $\exists$ map $[X/G] \longrightarrow \underbrace{X/G}_{\text{sch. quotient}}$ w/ nice properties.

This can be formalized beyond quotient stacks:

Def: Let $\mathfrak{X}$ be an alg. stack then $\mathfrak{X} \xrightarrow{\pi} \underset{\text{scheme}}{X}$ is a CMS if

(i) π is a bijection on geom. pts. ["Same underlying topology"]

(ii) π is initial among maps to schemes. [If $\mathfrak{X}$ is a moduli stack X approximates $\mathfrak{X}$].

Thm (Keel-Mori) If $\mathfrak{X}$ is a sep'd D.M. stack $\exists$ coarse moduli space $\mathfrak{X} \xrightarrow{\pi} X$.

E.g. $X = S$, $G \overset{\text{trivially}}{\curvearrowright} S$ then $[X/G] = BG$ is the stack of G-torsors.

This is algebraic. When G/S is fét $\Rightarrow$ BG is Deligne-Mumford.

$BG_S \to S$ is a CMS (stab everywhere

E.g. $[\mathbb{A}^1/(\mathbb{Z}/2\mathbb{Z})]$ when $\mathbb{Z}/2\mathbb{Z} \curvearrowright \mathbb{A}^1$, $x \mapsto x^2$, then $\operatorname{Spec} k \xrightarrow{x} \mathbb{A}^1$ $x, -x$
$\overset{\shortparallel}{x}$ $\downarrow$ $\quad$ $\mathrm{I} \downarrow$ $\searrow [\mathbb{A}^1/\mathbb{Z}/2\mathbb{Z}]$

$\mathrm{Stab}_0(x) = \mathbb{Z}/2\mathbb{Z}$. $\quad \operatorname{Spec} k[x] \to [\mathbb{A}^1/\mathbb{Z}/2] \xrightarrow{\text{cms}} \operatorname{Spec} k[x^2] = \mathbb{A}^1$] isom away from 0.

E.g. $[\mathbb{A}^2/\mathbb{Z}/2\mathbb{Z}]$ $\quad k[x,y] \to k[x,y]$, $x \mapsto -x$, $y \mapsto -y$ $\quad \mathbb{A}^2 \to \underbrace{[\mathbb{A}^2/\mathbb{Z}/2\mathbb{Z}]}_{\text{smooth.}} \xrightarrow{\text{cms.}} \operatorname{Spec} k[xy, y^2, x^2] \cong \dfrac{k[a,b,c]}{(c^2-ab)}$ } not smooth.

stabilizer only at origin.

E.g. Given X, $L \in \operatorname{Pic}(X)$, $s \in H^0(X,L)$, we can form the $\boxed{\text{root stack}}$ associated to (X, L, s)

$$\sqrt[n]{(X,L,s)} = \left\{ \left(T \to X, \underset{\operatorname{Pic}(X)}{\overset{\cap}{M}}, \sigma \in H^0(T,M), \phi\right) \;\middle|\; M^{\otimes n} \xrightarrow[\phi]{\sim} L,\ \sigma^{\otimes n} \longmapsto s \right\}.$$

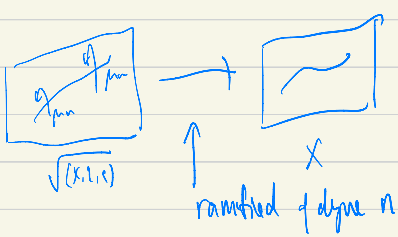
(M, σ, ϕ)
$\downarrow \gamma$
(M', σ', ϕ')
$\gamma: M \to M'$
$\gamma(\sigma) = \sigma'$
$M^{\otimes n} \xrightarrow{\gamma} M'^{\otimes n}$
$\phi \searrow \quad \swarrow \phi'$
L_T

s'pose $x: \operatorname{Spec} k \to X \setminus V(s)$ then

$k \to k$ must be identity b/c $\sigma \neq 0$.
$\sigma \mapsto \sigma$
$\sigma^n \quad \sigma^n$

elsewhere we have

$k \to k$, $0 \mapsto 0$ } no condition, so stabilizer here is μ_n everywhere



Thus a quasicoherent sheaf on BG $\leftrightarrow$ a representation of G

So <u>EX</u> i) $\mathrm{Pic}(B\mathbb{Z}/n\mathbb{Z})$ = Homs from $\mathbb{Z}/n\mathbb{Z} \to \mathbb{G}_m$ = nth roots of unity ii) $\mathrm{Pic}(B\mathbb{G}_m)$ = Hom's from $\mathbb{G}_m \to \mathbb{G}_m$, $z \mapsto z^n$ $= \mathbb{Z}$.

<u>E.g</u>: On $[X/G]$ a coherent sheaf is coherent sheaf F_X on X equipped w/ a G-action s.t. it is compatible w/ the action on X.

i.e. $\mathcal{F}$ is a G-equivariant sheaf. For a v.b. this can be simplified

a G-equivariant structure on a v.b. V on X is the "lift" of the action: $V \times G \xrightarrow{m_V} V$ s.t.

$$\begin{tikzcd} V \times G \arrow[r, "m_V"] \arrow[d, "(\pi, id)"'] \arrow[dr, phantom, "\circlearrowleft"] & V \arrow[d, "\pi"] \\ X \times G \arrow[r, "m_X"'] & X \end{tikzcd}$$

(i.e. $V \to X$ equivariant).

This is an instance of the $\boxed{\text{PRINCIPLE}}$ The geometry of $[X/G]$ is the G-equivariant geometry of X.

<u>E.g</u>: What is $\mathrm{Pic}([\mathbb{A}^1/\mathbb{Z}/2\mathbb{Z}])$? What are the $\mathbb{Z}/2\mathbb{Z}$-equivariant line bundles on $\mathbb{A}^1$? We have only

$$\begin{tikzcd} L = \mathbb{A}^1 \times \mathbb{A}^1 \arrow[r, "\tilde{\sigma}"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{A}^1 \times \mathbb{A}^1 \arrow[d, "p_1"] \\ \mathbb{A}^1 \arrow[r, "\sigma"] & \mathbb{A}^1 \end{tikzcd}$$

$(a,b) \longmapsto (-a, \tilde{\sigma}(b))$

& $\tilde{\sigma}$ must be linear so $\tilde{\sigma}(b)$ can be b or $-b$ when b then we get L_1 & $-b$ L_2.

$\boxed{L_1 = \mathcal{O}_{[X/G]}}$*: $\mathbb{A}^1 \times \mathbb{A}^1 \longrightarrow \mathbb{A}^1 \times \mathbb{A}^1$, $(a,b) \overset{id}{\mapsto} (a,b)$, $(a,b) \overset{\sigma}{\mapsto} (-a,b)$

L_2: $\mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1 \times \mathbb{A}^1$, $(a,b) \overset{id}{\longmapsto} (a,b)$, $(a,b) \overset{\sigma}{\longmapsto} (-a,-b)$

Observe that if V is a v.b. on $[X/G]$ & $\mathrm{Stab}(x)$ for $x \in X(\mathbb{C})$ then $\mathrm{Stab}(x)$ acts on V_x b/c the lift of $\sigma \in \mathrm{Stab}(x)$ to V

preserves the fiber V_x (by commutativity)

what happens here? $\mathbb{Z}/2\mathbb{Z} = \mathrm{Stab}(0) \curvearrowright (L_1)_0$ trivially but by

inversion on $(L_2)_0$. So $\mathrm{Pic}([\mathbb{A}^1/\mathbb{Z}/2\mathbb{Z}]) = \mathbb{Z}/2\mathbb{Z}$

$L_1 \leftrightarrow [\mathcal{O}_X] = \mathrm{id}$

$L_2 \leftrightarrow$ nontrivial elt.

$$\underset{\mathbb{A}^1}{\overset{\shortparallel}{\mathrm{Spec}\,k[x]}} \to [\mathbb{A}^1/\mathbb{Z}/2\mathbb{Z}] \xrightarrow{\pi} \mathrm{Spec}\,k[x^2]$$

$$\underbrace{}_{\pi}$$

It follows from G-S $[\mathbb{A}^1/(\mathbb{Z}/2\mathbb{Z})] \cong \overbrace{\sqrt{(\mathrm{Spec}\,k[x^2], k[x^2], x^2)}}^{\text{IS}}$ & in fact L_2 above is

the "universal" root of $\mathcal{O}_{\mathbb{A}^1_{x^2}}$.

Assume char$(k) \neq 2, 3$

Ex: $\mathbb{G}_m \curvearrowright \mathbb{A}^2$ $\lambda(a,b) = (\lambda^4 a, \lambda^6 b)$ & let $\Delta = V(4a^2 - 27b^3)$

λ^0 $(B\times\mathbb{A}^1)$ $(a,b,c) = (\lambda^4 a, \lambda^6 b, c)$

λ^{12} $[B\times\mathbb{A}^1]$ $(a,b,(4a^2-27b^3)c)$

$$\mathcal{M} = [\underbrace{\mathbb{A}^2 \setminus V(\Delta)}_{B}/\mathbb{G}_m] \to B\mathbb{G}_m$$

$$\begin{tikzcd} B \times \mathbb{A}^1 \arrow[r, "\mathbb{G}_m"] \arrow[d] & B \times \mathbb{A}^1 \arrow[d] \\ B \arrow[r] & \hat{B} \end{tikzcd}$$

$B \times \mathbb{A}^1 \times \mathbb{G}_m \to B \times \mathbb{A}^1$

$((a,b), c, \lambda) \mapsto ((\lambda^4 a, \lambda^6 b), \lambda c)$

What are the possible actions? Note that any action $/B$

induces a rep of $\mu_4 = \mathrm{Stab}(\underbrace{a, 0}_{\text{one orbit}})$ & $\mu_6 = \mathrm{Stab}(\underbrace{0, b}_{\text{one orbit}})$ &

$a \neq 0 \neq b$

$\mu_2 = \mathrm{Stab}(\underbrace{a, b}_{\text{the remaining orbits}})$ Given L we can look at

the induced rep of μ_4 & μ_6 to get a hom.

$$\begin{array}{ccc} & & 0 \\ & & \downarrow \\ & \phi \dashrightarrow & \mathbb{Z}/12\mathbb{Z} \\ & & \downarrow \\ \mathrm{Pic}(\mathcal{M}) & \to & \underset{\mathrm{Hom}(\mu_4, \mathbb{G}_m)}{\overset{\shortparallel}{\mathbb{Z}/4\mathbb{Z}}} \times \underset{\mathrm{Hom}(\mu_6, \mathbb{G}_m)}{\overset{\shortparallel}{\mathbb{Z}/6\mathbb{Z}}} \end{array}$$

But in fact there is a condition, the rep's

on μ_2 must all be the same b/c

$\mathrm{Hom}(\mu_{2,R}, \mathbb{G}_{m,R}) \cong (\mathbb{Z}/2\mathbb{Z})_R$ for any ring R $\Rightarrow$

$\downarrow$ restrict to μ_2

$\mathbb{Z}/2\mathbb{Z}$

The claim is that ϕ is an isomorphism. The action given above shows it is surjective. Why injective? Observe that every action $\phi : B \times \mathbb{A}^1 \times \mathbb{G}_m \to B \times \mathbb{A}^1$ of the form

$$((a,b), c, \lambda) \longmapsto ((\lambda^4 a, \lambda^6 b), \lambda^{f(a,b)}, \lambda)$$

where $f : B \to \underline{\mathrm{Hom}}(\mathbb{G}_m, \mathbb{G}_m)$ is defined by given $z \in B$ we have

$$\begin{tikzcd} \mathbb{A}^1 \times \mathbb{G}_m \arrow[d, "\wr"] & \\ \{z\} \times \mathbb{A}^1 \times \mathbb{G}_m \arrow[r, "\phi|"] \arrow[dr, "\text{linear action}"'] & B \times \mathbb{A}^1 \arrow[d, "p_2"] \\ & \mathbb{A}^1 \end{tikzcd}$$

so z defines a 1-dim'l rep of $\mathbb{G}_m$ i.e. an elt. of $\underline{\mathbb{Z}}$. B irred so constant.

Also we see that $B \times \mathbb{A}^1 \longrightarrow B \times \mathbb{A}^1 \quad ((a,b), c) \longmapsto ((a,b), (4a^3 + 27b^3)c)$ is an iso. equivariant from the trivial action to the λ^{12}-action. $\Rightarrow \mathrm{Pic}(\mathcal{M}) \cong \mathbb{Z}/12\mathbb{Z}$

<u>Punchline</u>: $\mathcal{M}_{1,1,\mathbb{Z}[1/6]} \cong \mathcal{M}$. <u>Idea</u>: Weierstrass form.

Rem. 1) $G \xrightarrow{\Delta} G \times G$ is rep'ble by algebraic spaces
$a \longmapsto (a, a, id)$
[$\Leftrightarrow$ $\forall T$ $\forall a, b \in G(T)$, $\underline{Isom}_T(a,b) \to T$ is rep'ble by an alg. space/T]

2) $\exists$ smooth surjection $U \xrightarrow[\text{smooth}]{\phi} G$] (U, ϕ) is called an atlas of G

~~Examples~~: - ~~Any~~ $\exists$ ~~étale~~ scheme $U \xrightarrow{\phi} G$ an G ~~is~~ algebraic ~~Deligne-Mumford space~~.

Idea: Same as manifolds, the local geometry of F or G can be studied on U!

If ~~finite~~ group $G \curvearrowright X$ freely] scheme then the quotient exists as an algebraic space & $X \to X/G$ is étale. [Closed under finite quotients]

- Let $D \subseteq X$ be a Cartier divisor s.t. $N_{D/X} = \mathcal{O}_X(D)|_D$ is antiample (over smooth schemes $/k = \bar{k}$), then $\exists$ a contraction in the category of alg. spaces i.e. $\exists \bar{X}$ alg. space

$$\begin{tikzcd}
D \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d] \arrow[dr, phantom, "\square"] & X \setminus D \arrow[l] \arrow[d] \\
\operatorname{Spec} k \arrow[r, "p"] & \bar{X} & \bar{X} \setminus p \arrow[l]
\end{tikzcd}$$

- If $X \xrightarrow{f} S$ proper + flat (+ f.p.) then $\mathrm{Hilb}_{X/S} \to S$ is rep'ble [closed under Hilbert "schemes"]

Examples of algebraic stacks

1 (i) $BGL_n = \mathrm{Vect}_n$ $\quad T \longmapsto$ grpd. of vect. bundles of rk n on T

Pf: Need to show $I = \underline{Isom}_T(V_1, V_2) \to T$ is rep'ble by a scheme $\forall T$ & $\forall V_1, V_2 \in BGL_n(T)$. But $I \to T$ is Zar. locally on T $\cong \mathrm{Isom}(\mathcal{O}^{\oplus n}, \mathcal{O}^{\oplus n})$ $\cong GL_n$ & scheme.

$\Rightarrow$ $\Delta : BGL_n \to BGL_n \times BGL_n$ is rep'ble by affine schemes. ✓

(ii) Is there a smooth atlas?

$$\begin{tikzcd}
I = \underline{Isom}(\mathcal{O}^{\oplus n}, V) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \operatorname{Spec} \mathbb{Z} \arrow[d, "\phi \text{ trivial } \mathcal{O}^{\oplus n}"] \\
T \arrow[r, "V"] & BGL_n
\end{tikzcd}$$

$I \to T$ is (everywhere) Zar. locally on T iso. to GL_n. $\Rightarrow$ since GL_n is smooth $\Rightarrow$ ϕ is a smooth cover