

The Mittag-Leffler property & Moduli

Def: A sequence of sets (A_n) is Mittag-Leffler if $\forall n \exists N \geq n$ s.t.
 $\text{Im}[A_m \rightarrow A_n] = \text{Im}[A_N \rightarrow A_n] \quad \forall m \geq N$

mixed char.

Fix a complete DVR $\text{Spec } R$, the goal of this talk

Main Thm (Kresch M.) Let \mathcal{X} of finite type be a map of stacks s.t.
 \downarrow finite type $\text{Spec } R$ \downarrow smooth $\text{Spec } R$

$([\mathcal{X}(R/m^{n+1})])$ is a Mittag-Leffler sequence.

Rem: Compare w/ a smooth morphism $\mathcal{X} \rightarrow \text{Spec } R$ Formal smoothness

$\Rightarrow ([\mathcal{X}(R/m^{n+1})])$ has surjective connecting maps.

Cor: If $X \rightarrow \text{Spec } R$ is proper + flat map of schemes, then $(\text{Pic}^0(X_n))$ is

ML.

Pf: The CFG $T \mapsto \{L \in \text{Pic}^0(X \times_R T)\}$ is an algebraic stack of f.l. over $\text{Spec } R$, $\mathcal{P}ic_{X_n/k(R)}^0 \rightarrow \text{Spk}(R)$ is smooth

b/c it factors $\text{Pic}_{X_n/k(R)}^0 \rightarrow \text{Pic}_{X_n/k(R)} \rightarrow \text{Spk}(n)$
 \uparrow since so smooth. \downarrow smooth by Cartier's lem (groups in char 0 are smooth)

so apply main thm. \blacksquare

Key ingredient for main thm is the following black box:

Elkik: Given $\text{Spec } B = X$
 \downarrow f.t.
 $\text{Spec } R$ be a map of schemes, then $\exists n_0, r > 0$ s.t.
 s.t. $X_n \rightarrow \text{Spec } (R/m^{n+r})$ smooth.

if $\exists \text{Spec } R \xrightarrow{\xi_n} X$ for $n > n_0$ then $\exists \text{Spec } R \xrightarrow{\xi} X$ s.t.

$\xi \equiv \xi_n \pmod{m^{n-r}}$. In particular, $(X(R/m^n))$ is M.L.

(for any $n > 0$ if $\xi_n \in X(R/m^n)$ extends to $\xi_{n+r} \in X(R/m^{n+r})$ it extends to

a $\xi \in X(R)$ s.t. $\xi \equiv \xi_{n+r} \pmod{m^{n+r}}$.)

How to upgrade this to stacks? For any spaces $X \exists \text{Spec } R \rightarrow X$
 \downarrow smooth

s.t. every field-valued pt lifts (due to Knutson). So we have

(by formal smoothness) surjectives $\text{Spec } B(A) \rightarrow X(A)$ for all

local Artinian rmp. So if $\text{Spec } B \xrightarrow{\pi} X \rightarrow \text{Spec } R$ we can set
 $\xrightarrow{\text{gen. smooth}}$

that $(X(R/m^{n+r}))$ is ML since $(\text{Spec } B(R/m^{n+r}))$ is since

$$\pi(\text{Im}[\text{Spec } B(R/m^n) \rightarrow \text{Spec } B(R/m^i)]) = \text{Im}[X(R/m^n) \rightarrow X(R/m^i)].$$

So RHS stabilizes b/c LHS does.

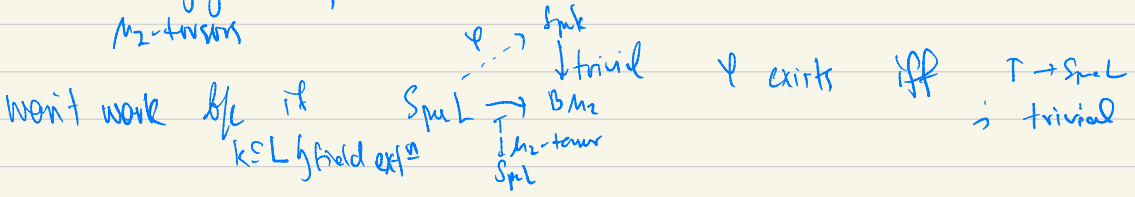
Same arg. will work if we can show \exists smooth cover
 $X \rightarrow \mathcal{E}$ s.t. every field-valued pt of \mathcal{E} lifts.

Does this exist?

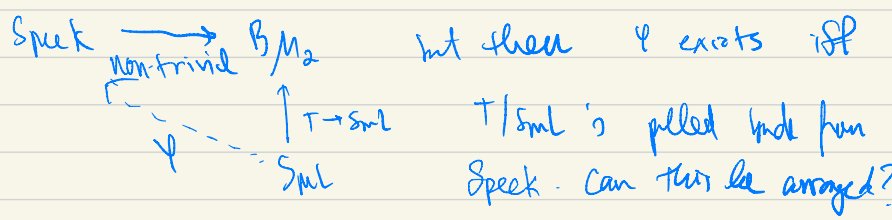
This will be the focus of the remainder of the talk

This is non-trivial, as illustrated by the following example

Ex: $\mathcal{X} = B\mu_2 \rightarrow \text{Spec } k$. The standard atlas $\text{Spec } k \rightarrow B\mu_2$
 classifying stack of μ_2 -torsors



Next try?



$$0 \rightarrow \mu_2 \rightarrow \mathbb{G}_m \xrightarrow{x^2} \mathbb{G}_m \rightarrow 0 \quad \text{coker} \rightarrow \quad 0 \rightarrow H^0(\text{Spec } k, \mu_2) \rightarrow L^x \xrightarrow{x^2} L^x \rightarrow H^1(\text{Spec } k, \mu_2)$$

$$\downarrow \\
 H^2(\text{Spec } k, \mathbb{G}_m) \\
 \text{" HT 90. } \\
 0$$

This tells us

$$H^1(\text{Spec } k, \mu_2) = L^x / L^{x^2}$$

" μ_2 -torsors over $\text{Spec } k$ " \cong

So no field will work

$\text{Spec } k \xrightarrow{x} B\mu_2$ $x \in L^x$ then $\exists L(\frac{x}{z}) \geq L$ $\{ z \in L(\frac{x}{z})^x$
 $\psi \dashrightarrow \uparrow z$ $\text{Spec } L(z)$ $\{ \psi \text{ exists when } x^{-1}z \in L(z)^{x^2}$ for $x \in L$
 which is clearly not true b/c this would
 imply in $L(\frac{x}{z})(z)$ z is a square \downarrow

So how? $A'_z \xrightarrow{z} B_{mz}$ then if $\uparrow \text{rel } L/Lz$ \exists map $\text{Sp} L \rightarrow A'_z \xrightarrow{z} B_{mz}$
 $l \leftarrow z$

m this is the desired cover.

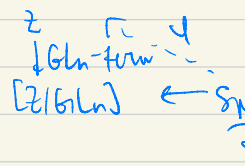
The fact this is always true.

Thm (Lauvmon-Monnet-Bailly, Pirisi, Deshmukh) 1) Given $\mathcal{X} \rightarrow S$ alg. stack. \exists smooth cover $X_d \rightarrow \mathcal{X}$ s.t. every field-valued pt $\overset{\text{def}}{\text{of}} \mathcal{X}$ lifts. \exists where $X_d \rightarrow S$ f.t.

2) If S is Noeth. \mathcal{X}/S f.t. \exists \mathcal{X} has affine stabilizers, then \exists

f.t. $X \rightarrow \mathcal{X}$ s.t. every field-valued pt. lifts.

Step #1: If

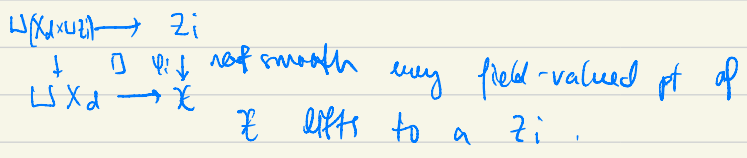


Sketch of 2) assuming 1)

then \forall exists b/c $\text{Sp} \mathcal{O}$ has no nontrivial GL_n -torsors.

Step #2 (Knech) n f.t. patches w/ aff. stab. are stratified by global quotient stacks. \exists $U_0 \subseteq U_1 \subseteq \dots \subseteq U_n = \mathcal{X}$ s.t. $U_i \setminus U_{i-1} \cong [Z_i / \text{GL}_n]$ alg. que.

Step #3: Use Z_i to show only finitely many X_d are required.



$Z_i \times \mathbb{A}^1 \times \mathbb{A}^1 \xrightarrow{b_i} Z_i$ every field valued pt. lifts, so true for gen. pts

so \exists section $\underbrace{U_i \cap Z_i}_{\text{dense}} \rightarrow \mathbb{A}^1 \times \mathbb{A}^1$, i.e. $U_i \subseteq \mathbb{A}^1 \times \mathbb{A}^1 \times Z_i$ for some $d_i > 0$.

Repeat w/ gen. pts of $Z_i \cap U_i$, this terminates after finite many steps by Noetherianity. Same w/ all $Z_i \hookrightarrow Z_{i+1}$. This gives

$$X = \varprojlim_{d \in \mathbb{N}} X_d \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{\quad} \end{array} X \quad \text{s.t. for all } i \in \mathbb{N} \exists \text{ lift.}$$

Z_i
 \downarrow

$\Rightarrow X \rightarrow X$ is the dense cover. \square