The Mittag-Leffler property & Moduli

Def: A sequence of sets (A_n) is Mittag-Leffler if $\forall n\ \exists N \geq n$ s.t.
$$\operatorname{Im}[A_m \to A_n] = \operatorname{Im}[A_N \to A_n] \quad \forall m \geq N,$$

Fix a complete mixed char. dvr Spec R, the goal of this talk

Main Thm (Kresch–M.) Let $\mathcal{X}$ (affine stabilizers) $\downarrow$ finite type Spec R be a map of stacks s.t. $\mathcal{X}_k \downarrow$ smooth $\operatorname{Spec} k(R)$

$\{[\mathcal{X}(R/m^{n+1})]\}$ is a Mittag-Leffler sequence.

Rem: Compare w/ a smooth morphism
$$\begin{tikzcd} \mathcal{X} \arrow[d] \\ \operatorname{Spec} R \end{tikzcd}$$
Formal smoothness

$\Rightarrow ([\mathcal{X}(R/m^{n+1})])$ has surjective connecting maps.

Cor: If $X \xrightarrow{f} \operatorname{Spec} R$ is proper + flat map of schemes, then $(\operatorname{Pic}^0(X_n))$ is ML.

Pf: The CFG $T \mapsto \{L \in \operatorname{Pic}^0(X \times_R T)\}$ ← groupoid ✱ (small lie, should be Pic^τ) is an algebraic stack of f.t. over Spec R, & $\underline{\operatorname{Pic}}^0_{X_k/k(R)} \to \operatorname{Spec} k(R)$ is smooth b/c it factors

$$\begin{tikzcd} \underline{\operatorname{Pic}}^0_{X_k/k(R)} \arrow[r] & \underline{\operatorname{Pic}}_{X_k/k(R)} \arrow[r] & \operatorname{Spec} k(R) \end{tikzcd}$$

↑ open so smooth. — smooth by Cartier's thm (groups in char 0 are smooth)

So apply main thm. ∎

Key ingredient for main thm is the following black box:

$\{\operatorname{Spec} B(R/m^{n+1})\}$ is M.L.

$\boxed{\text{Elkik}}$: Given
$$\begin{tikzcd} \operatorname{Spec} B = X \arrow[d, "f.t."] \\ \operatorname{Spec} R \end{tikzcd}$$
be a map of schemes, s.t. $X_\eta \to \operatorname{Spec} K(R)$ smooth. then $\exists\ n_0, r > 0$ s.t.

if $\exists\ \operatorname{Spec} R/m^n \xrightarrow{\xi_n} X$ for $n > n_0$ then $\exists\ \operatorname{Spec} R \xrightarrow{\xi} X$ s.t.

$\xi \equiv \xi_n \bmod m^{n-r}$. In particular, $(X(R/m^n))$ is M.L.

(for any $n > 0$ if $\xi_m \in X(R/m^n)$ extends to $\xi_{big} \in X(R/m^{n+n_0+r})$ it extends to a $\xi \in X(R)$ s.t. $\xi \equiv \xi_{big} \bmod m^{n+n_0}$.)

How to upgrade this to stacks? For alg. spaces X $\exists\ \operatorname{Spec} B \xrightarrow[\text{smooth}]{} X$ s.t. every field-valued pt lifts (due to Knutson). So we have (by formal smoothness) surjections $\operatorname{Spec} B(A) \twoheadrightarrow X(A)$ for all local Artian rings. So if
$$\begin{tikzcd} \operatorname{Spec} B \arrow[r, "\pi"] \arrow[rr, bend right, "\text{gen. smooth}"'] & X \arrow[r] & \operatorname{Spec} R \end{tikzcd}$$
we can set that $(X(R/m^{n+1}))$ is ML since $(\operatorname{Spec} B(R/m^{n+1}))$ is since

$$\pi(\operatorname{Im}[\operatorname{Spec} B(R/m^\ell) \to \operatorname{Spec} B(R/m^i)]) = \operatorname{Im}[X(R/m^\ell) \to X(R/m^i)].$$

So RHS stabilizes b/c LHS does.

Same arg. will work if we can show $\exists$ smooth cover $X \to \mathfrak{X}$ s.t. every field-valued pt of $\mathfrak{X}$ lifts.

Does this exist?

This ? will be the focus of the remainder of the talk

This is non-trivial, as illustrated by the following example

<u>Ex:</u> $\mathcal{X} = B\mu_2 \to \operatorname{Spec} k$. The standard atlas $\operatorname{Spec} k \underset{\substack{\text{trivial}\\ \mu_2\text{-torsor}}}{\to} B\mu_2$

$B\mu_2$ $\overset{\shortparallel}{}$ classifying stack of μ_2-torsors

won't work b/c if $k \subseteq L$ field ext$^{\underline{n}}$

$$\begin{tikzcd}
 & \operatorname{Spec} k \arrow[d, "\text{trivial}"] \\
\operatorname{Spec} L \arrow[ur, dashed, "\varphi"] \arrow[r, "T"] & B\mu_2
\end{tikzcd}$$

T $\downarrow$ μ_2-torsor $\operatorname{Spec} L$

φ exists iff $T \to \operatorname{Spec} L$ is trivial

Next try?

$$\begin{tikzcd}
\operatorname{Spec} k \arrow[r, "\text{non-trivial}"'] & B\mu_2 \\
 & \operatorname{Spec} L \arrow[u, "T\to\operatorname{Spec} L"'] \arrow[ul, dashed, "\varphi"]
\end{tikzcd}$$

but then φ exists iff $T/\operatorname{Spec} L$ is pulled back from $\operatorname{Spec} k$. Can this be arranged?

$$0 \to \mu_2 \to \mathbb{G}_{m,L} \xrightarrow{\times 2} \mathbb{G}_{m,L} \to 0 \overset{\text{cohs.}}{\rightsquigarrow} 0 \to H^0(\operatorname{Spec} L, \mu_2) \to L^\times \xrightarrow{\times 2} L^\times \to H^1(\operatorname{Spec} L, \mu_2)$$

$$\downarrow$$

$$H^2(\operatorname{Spec} L, \mathbb{G}_m) \overset{HT90.}{=} 0$$

This tells us $H^1(\operatorname{Spec} L, \mu_2) = L^\times / L^{\times 2}$

$$\shortparallel$$

$\{\mu_2\text{-torsors over } \operatorname{Spec} L\} / \simeq$

So no field will work

$$\begin{tikzcd}
\operatorname{Spec} k \arrow[r, "x"] & B\mu_2 \\
 & \operatorname{Spec} L(z) \arrow[u, "z"'] \arrow[ul, dashed, "\varphi"]
\end{tikzcd}$$

$x \in L^\times$ then $\exists\, L(\overset{\text{indeterminate}}{z}) \supseteq L$ & $z \in \dfrac{L(z)^\times}{L(z)^{\times 2}}$

& φ exists when $x^{-1}z \in L(z)^{\times 2}$ for $x \in L$

which is clearly not true b/c this would imply in $L(\sqrt{x})(z)$ z is a square ↯

So how ? $\uparrow$ then if

$$\begin{tikzcd}
\mathbb{A}^1_z \setminus \{0\} \arrow[r, "z"] & B\mu_2 \\
& \operatorname{Spec} L \arrow[u, "\ell \in L^\times/L^{\times 2}"']
\end{tikzcd}$$

$\exists$ map $\operatorname{Spec} L \longrightarrow \mathbb{A}^1_z \setminus \{0\}$

$\ell \longleftarrow\!\shortmid z$

so this is the desired cover.

In fact this is always true.

Thm (Laumon-Moret-Bailly, Pirisi, Deshmukh) Given $\mathcal{X} \to S$ alg. stack. 1) $\exists$ smooth cover $\bigsqcup_{d \in \mathbb{N}} X_d \to \mathcal{X}$ s.t. every field-valued pt of $\mathcal{X}$ lifts. & where $X_d \to S$ f.t.

2) If S is Noeth. $\mathcal{X}/S$ f.t. & $\mathcal{X}$ has affine stabilizers, then $\exists$ f.t. $X \to \mathcal{X}$ s.t. every field-valued pt. lifts.

Sketch of 2) assuming 1)

Step #1: If

$$\begin{tikzcd}
Z \arrow[d, "GL_n\text{-torsor}"'] & \\
{[Z/GL_n]} & \operatorname{Spec} \mathcal{O} \arrow[l] \arrow[ul, dashed, "\psi"']
\end{tikzcd}$$

$\underbrace{\operatorname{Spec} \mathcal{O}}_{\text{semi-local}}$ then ψ exists b/c $\operatorname{Spec} \mathcal{O}$ has no non-trivial GL_n-torsors.

Step #2 (Kresch) "f.t. stacks w/ aff. stab. are stratified by global quotient stacks" $\exists$ $\mathcal{U}_0 \subseteq \mathcal{U}_1 \subseteq \dots \subseteq \mathcal{U}_n = \mathcal{X}$ s.t. $\mathcal{U}_i \setminus \mathcal{U}_{i-1} \cong [\underbrace{Z_i}_{\text{alg. space}}/GL_{n_i}]$.

Step #3: Use Z_i to show only finitely many X_d are required.

$$\begin{tikzcd}
\bigsqcup(X_d \times \sqcup Z_i) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Z_i \arrow[d, "\varphi_i"] \\
\bigsqcup X_d \arrow[r] & \mathcal{X}
\end{tikzcd}$$

not smooth every field-valued pt of $\mathcal{X}$ lifts to a Z_i.

$Z_i \times \bigsqcup X_d \xrightarrow{b_i} Z_i$ every field valued pt. lifts, so true for gen. pts

so $\exists$ section $\underbrace{U_i \subseteq Z_i}_{\text{dense}} \to \bigsqcup X_d$, i.e. $U_i \subseteq \bigsqcup_{d \le d'} X_d \times Z_i$ for some $d' > 0$.

Repeat w/ gen. pts of $Z_i \setminus U_i$, this terminates after finite many steps by Noetherianity. Same w/ all $Z_1, \dots, Z_n$. This gives

$$\begin{tikzcd} & Z_i \arrow[dl, dashed] \arrow[d] \\ X = \bigsqcup_{d \le D} X_d \arrow[r] & \mathfrak{X} \end{tikzcd}$$

s.t. for all $i = 1, \dots, n$ $\exists$ lift.

$\Rightarrow$ $X \to \mathfrak{X}$ is the dsrd cover. $\blacksquare$