Very roughly I often think of a stack as a scheme along w/ groups attached at pts. The following justifies this 6

Thm (Gerashenko–Satriano) Let $\mathcal{X}$ be a smooth D.M. stack f.t./k w/ trivial generic stabilizers (let $\mathcal{X} \xrightarrow{\pi} X$ be the cms) & let $D \subseteq X$ be the ramification divisor $D = \underbrace{D_1}_{\deg e_1} \cup \cdots \cup \underbrace{D_n}_{\deg e_n}$ then this uniquely determines $\mathcal{X}$

(i.e. $\mathcal{X} \cong \sqrt{(X^{can}, L, s)} \to \sqrt{(X^{can}, L, s)} \to X^{can} \to X$)

The Picard group of $\mathcal{M}_{1,1}$

What is a q.coherent sheaf on a D.M. stack X? Fix $U \xrightarrow{et} X$

Def: A q.coh sheaf on X, is a descent datum $(F_U, p_1^* F_U \xrightarrow{\phi} p_2^* F_U)$ of q.coh sheaves. It is coherent if F_U is.

$U \times_X U \rightrightarrows U \to X$

Ex: Consider G finite, then use the étale atlas & we have from last time

$$\begin{array}{ccc} G = \operatorname{Spec} k \times_{BG} \operatorname{Spec} k & \longrightarrow & \operatorname{Spec} k \\ \downarrow p_1 & & \downarrow \text{trivial} \\ \operatorname{Spec} k & \xrightarrow{\text{trivial}} & BG \end{array}$$

& $\operatorname{Spec} k \times_{BG} \operatorname{Spec} k \times_{BG} \operatorname{Spec} k \cong G \times G$

$$\begin{array}{ccc} \operatorname{Spec} k \times_{BG} \operatorname{Spec} k \times_{BG} \operatorname{Spec} k & \cong & G \times G \\ \downarrow p_{12} \ \downarrow p_{13} \ \downarrow p_{23} & & p_1 \downarrow \ \downarrow m \ \downarrow p_2 \\ \operatorname{Spec} k \times \operatorname{Spec} k & \cong & G \times G \end{array}$$

Now you can check that a qcoh module on BG $\iff$ is the same as a k-vect. space w/ $|G|$ auts (ϕ_g) satisfying the cocycle condition i.e. $\phi_{gh} = \phi_h \circ \phi_g$ i.e. $G \to \operatorname{Aut}_k(V)$, $g \mapsto \phi_g$; a grp hom.

Thus a quasicoherent sheaf on BG $\leftrightarrow$ a representation of G

So <u>EX</u> i) $\mathrm{Pic}(B\mathbb{Z}/n\mathbb{Z})$ = Homs from $\mathbb{Z}/n\mathbb{Z} \to \mathbb{G}_m$ = nth roots of unity ii) $\mathrm{Pic}(B\mathbb{G}_m)$ = Hom's from $\mathbb{G}_m \to \mathbb{G}_m$ $z \mapsto z^n$ $= \mathbb{Z}$.

<u>E.g</u>: On $[X/G]$ a coherent sheaf is coherent sheaf F_X on X equipped w/ a G-action s.t. it's compatible w/ the action on X.

i.e. $\mathcal{F}$ is a G-equivariant sheaf. For a v.b. this can be simplified

a G-equivariant structure on a v.b. V on X is the "lift" of the action: $V \times G \xrightarrow{m_V} V$ s.t.

$$\begin{tikzcd} V \times G \arrow[r, "m_V"] \arrow[d, "(\pi, id)"'] \arrow[dr, phantom, "\circlearrowleft"] & V \arrow[d, "\pi"] \\ X \times G \arrow[r, "m_X"'] & X \end{tikzcd}$$

(i.e. $V \to X$ equivariant)

This is an instance of the $\boxed{\text{PRINCIPLE}}$ The geometry of $[X/G]$ is the G-equivariant geometry of X.

<u>E.g</u>: What is $\mathrm{Pic}([\mathbb{A}^1/\mathbb{Z}/2\mathbb{Z}])$? What are the $\mathbb{Z}/2\mathbb{Z}$-equivariant line bundles on $\mathbb{A}^1$? We have only $L \cong$

$$\begin{tikzcd} \mathbb{A}^1 \times \mathbb{A}^1 \arrow[r, "\tilde{\sigma}"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{A}^1 \times \mathbb{A}^1 \arrow[d, "p_1"] \\ \mathbb{A}^1 \arrow[r, "\sigma"'] & \mathbb{A}^1 \end{tikzcd}$$

$(a,b) \longmapsto (-a, \tilde{\sigma}(b))$

& $\tilde{\sigma}$ must be linear so $\tilde{\sigma}(b)$ can be b or $-b$ when b then we get L_1 & $-b$ L_2.

$\boxed{L_1 = \mathcal{O}_{[X/G]}}$ *

$\mathbb{A}^1 \times \mathbb{A}^1 \longrightarrow \mathbb{A}^1 \times \mathbb{A}^1$
$(a,b) \overset{id}{\mapsto} (a,b)$
$(a,b) \overset{\sigma}{\mapsto} (-a,b)$

L_2
$\mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1 \times \mathbb{A}^1$
$(a,b) \overset{id}{\longmapsto} (a,b)$
$(a,b) \overset{\sigma}{\longmapsto} (-a,-b)$

Observe that if V is a v.b. on $[X/G]$ & $\mathrm{Stab}(x)$ for $x \in X(\mathbb{C})$ then $\mathrm{Stab}(x)$ acts on V_x b/c the lift of $\sigma \in \mathrm{Stab}(x)$ to V

preserves the fiber V_x (by commutativity)

what happens here? $\mathbb{Z}/2\mathbb{Z} = \mathrm{Stab}(0) \overset{\text{trivially}}{\curvearrowright} (L_1)_0$ but by inversion on $(L_2)_0$. So $\mathrm{Pic}([\mathbb{A}^1/\mathbb{Z}/2\mathbb{Z}]) = \mathbb{Z}/2\mathbb{Z}$

$L_1 \leftrightarrow [\mathcal{O}_X] = \mathrm{id}$

$L_2 \leftrightarrow$ nontrivial elt.

$$\underset{\mathbb{A}^1}{\overset{\shortparallel}{\mathrm{Spec}\, k[x]}} \to [\mathbb{A}^1/\mathbb{Z}/2\mathbb{Z}] \xrightarrow{\pi} \mathrm{Spec}\, k[x^2]$$

It follows from G-S $[\mathbb{A}^1/(\mathbb{Z}/2\mathbb{Z})] \cong \sqrt{(\mathrm{Spec}\, k[x^2], k[x^2], x^2)}$ & in fact L_2 above is the "universal" root of $\mathcal{O}_{\mathbb{A}^1_{x^2}}$.

Assume $\mathrm{char}(k) \neq 2, 3$

Ex: $\mathbb{G}_m \curvearrowright \mathbb{A}^2$ $\lambda(a,b) = (\lambda^4 a, \lambda^6 b)$ & let $\Delta = V(4a^2 - 27b^3)$

λ^0 $(B\times\mathbb{A}^1)$ $(a,b,c) = (\lambda^4 a, \lambda^6 b, c)$

λ^{12} $[B\times\mathbb{A}^1]$ $(a,b,(4a^2-27b^3)c)$

$$\mathcal{M} = [\underbrace{\mathbb{A}^2 \setminus V(\Delta)}_{B} / \mathbb{G}_m] \to B\mathbb{G}_m$$

$$\begin{tikzcd} B \times \mathbb{A}^1 \arrow[r, "\mathbb{G}_m"] \arrow[d] & B \times \mathbb{A}^1 \arrow[d] \\ B \arrow[r] & B \end{tikzcd}$$

$B \times \mathbb{A}^1 \times \mathbb{G}_m \longrightarrow B \times \mathbb{A}^1$

$((a,b), c, \lambda) \mapsto ((\lambda^4 a, \lambda^6 b), \lambda c)$

What are the possible actions? Note that any action / B induces a rep of $\mu_4 = \mathrm{Stab}(\underbrace{a,0}_{\text{one orbit}})$ & $\mu_6 = \mathrm{Stab}(\underbrace{0,b}_{\text{one orbit}})$ &

$a \neq 0 \neq b$

$\mu_2 = \mathrm{Stab}(\underbrace{a,b}_{\text{the remaining orbits}})$ Given L we can look at the induced rep of μ_4 & μ_6 to get a hom.

$$\begin{array}{c} 0 \\ \downarrow \\ \phi \dashrightarrow \mathbb{Z}/12\mathbb{Z} \\ \downarrow \end{array}$$

$$\mathrm{Pic}(\mathcal{M}) \to \underset{\mathrm{Hom}(\mu_4, \mathbb{G}_m)}{\overset{\shortparallel}{\mathbb{Z}/4\mathbb{Z}}} \times \underset{\mathrm{Hom}(\mu_6, \mathbb{G}_m)}{\overset{\shortparallel}{\mathbb{Z}/6\mathbb{Z}}}$$

But in fact there is a condition, the rep's on μ_2 must all be the same b/c

$\downarrow$ restrict to μ_2

$\mathbb{Z}/2\mathbb{Z}$

$\Rightarrow$

$\mathrm{Hom}(\mu_{2,R}, \mathbb{G}_{m,R}) \cong (\mathbb{Z}/2\mathbb{Z})_R$ for any ring R

The claim is that ϕ is an isomorphism. The action given above shows it's surjective. Why injective? Observe that every action $\phi : B \times \mathbb{A}^1 \times \mathbb{G}_m \to B \times \mathbb{A}^1$ of the form

$$((a,b), c, \lambda) \longmapsto ((\lambda^4 a, \lambda^6 b), \lambda^{f(a,b)}, \lambda)$$

where $f : B \to \underline{\mathrm{Hom}}(\mathbb{G}_m, \mathbb{G}_m)$ is defined by given $z \in B$ we have

$$\begin{tikzcd} \mathbb{A}^1 \times \mathbb{G}_m \arrow[d, "\wr"] & \\ \{z\} \times \mathbb{A}^1 \times \mathbb{G}_m \arrow[r, "\phi|"] \arrow[dr, "\text{linear action}"'] & B \times \mathbb{A}^1 \arrow[d, "p_2"] \\ & \mathbb{A}^1 \end{tikzcd}$$

so z defines a 1-dim'l rep of $\mathbb{G}_m$ i.e. an elt. of $\underline{\mathbb{Z}}$. B irred so constant.

Also we see that $B \times \mathbb{A}^1 \longrightarrow B \times \mathbb{A}^1$ $((a,b), c) \longmapsto ((a,b), (4a^3 + 27b^3) c)$ is an iso. equivariant for the trivial action to the λ^{12}-action. $\Rightarrow \mathrm{Pic}(\mathcal{M}) \cong \mathbb{Z}/12\mathbb{Z}$

<u>Punchline</u>: $\mathcal{M}_{1,1,\mathbb{Z}[1/6]} \cong \mathcal{M}$. <u>Idea</u>: Weierstrass form.