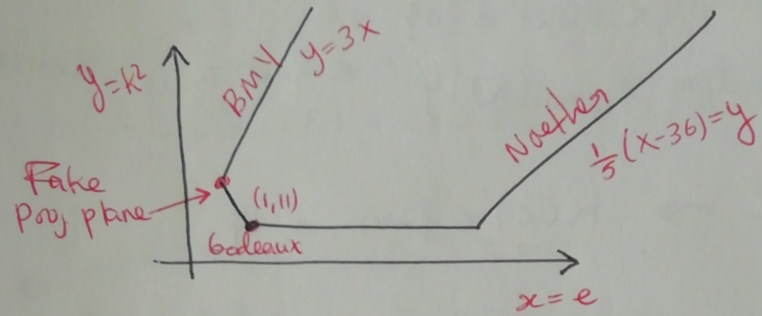
Geografía de superficies / $\mathbb{C}$

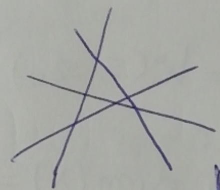
①



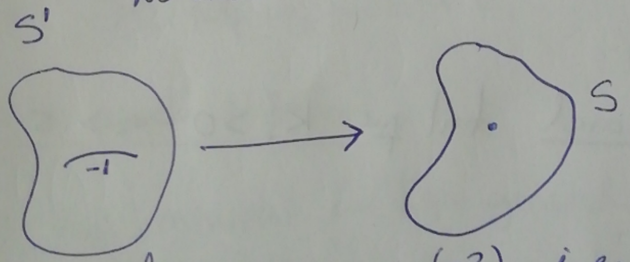
① ¿Qué es geografía?

② Resultado (Persson 1981)

③ Cubrimientos cíclicos

④ Ejemplos, $\{X_p\}_{p\geq 3}$  $\subseteq \mathbb{P}^2$

<u>Recordar.</u> S sup. minimal si no posee curvas (-1), i.e., No existe $E\subseteq S$ tal que $E\simeq \mathbb{P}^1$, con $E^2=-1$.



S podría tener curvas (-2), i.e., $E\simeq\mathbb{P}^1$, $E^2=-2$.

Notar que: $2g(E)-2 = E^2 + K_S\cdot E \rightarrow K_S E = 0$

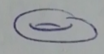
Nakai-Moishezon: L amplio $\longleftrightarrow$ $L^2>0$, $L\cdot C>0$ $\forall C\subseteq S$ curva.

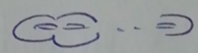
Dimensión de Kodaira : X var. proyectiva suave $/\mathbb{C}$.

$$\mathrm{Kod}(X) = \begin{cases} -\infty, & H^0(X, nK_X) = \{0\} \\ \max\{\dim(\varphi_{|nK_X|}(X))\} \end{cases} \in \{-\infty, 0, 1, \dots, \dim(X)\}$$

E.g. $X =$ curva $C \rightsquigarrow K(C) \in \{-\infty, 0, 1\}$

$K(C) = -\infty \iff g(C) = 0 \iff C \simeq \mathbb{P}^1$

$\mathrm{Kod}(C) = 0 \iff g(C) = 1 \iff C \simeq$ 

$\mathrm{Kod}(C) = 1 \iff g(C) \geq 2 \iff C \simeq$ 

tipo general.

$p_g(S) = h^0(S, \Omega^2_S)$, $q(S) = h^0(S, \Omega^1_S)$.

Def: X tipo general si $\dim(X) = \mathrm{Kod}(X)$

Una sup. S es de tipo minimal general si $\mathrm{Kod}(X) = 2$, i.e., K_S big & nef

Teo: S superficie no racional tal que $K_S^2 > 0 \iff S$ de tipo general.

S sup minimal $/ \mathbb{C}$

$$-\infty \begin{cases} \to \mathbb{P}^2, \quad \mathbb{F}_n := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n)) \\ \to \mathbb{P}^2_C(\mathcal{E}), \quad C \not\cong \mathbb{P}^1, \quad rg(\mathcal{E}) = 2. \end{cases}$$

$$0 \begin{cases} \to \text{Enriques}, \quad p_g = q = 0 \\ \to \text{Bielíptica}, \quad p_g = 0,\ q = 1 \\ \to K3, \quad p_g = 1,\ q = 0 \\ \to \text{Abelianas}, \quad p_g = 1,\ q = 2. \end{cases}$$

$1 \Rightarrow$ Fibración elíptica, $p_g, q \geq 0$.

$2 \rightsquigarrow$ tipo general, $p_g, q \geq 0$.

Noether (Max) $12\chi = K^2 + e$, $e = \chi_{top}(S)$, $\chi = \chi(\mathcal{O}_S)$.

i.e., $12(1 - q + p_g) = K^2 + e$.

- Receta de Mumford.

S Sup. minimal de tipo general

$$P_S(m) = \chi(S, mK_S) \underset{R\text{-}R}{=} \frac{m(m-1)}{2} K_S^2 + \chi(\mathcal{O}_S)$$

Fijar K^2_S y $\chi(\mathcal{O}_S) \Longleftrightarrow$ Fijar K^2_S y $e(S)$

(Gieseke 1977) $M_{K^2,\chi}$ Variedad quasi-projectiva.

$$\left[S_{can} \text{ con } K^2_{can} = K^2,\ \chi(\mathcal{O}_{S_{can}}) = \chi\right]$$

$\rightsquigarrow M_{K^2,e}$.

¿Dados $(K^2, e) \in \mathbb{Z}\times\mathbb{Z}$ es $M_{K^2,e} \neq \phi$?

- $K^2 + e \equiv 0 \mod 12$
- Castelnuovo $K^2, e > 0$
- Noether $\frac{1}{5}(e-36) \leq K^2$
- Bogomolov - Miyaoka - Yau. $K^2 \leq 3e$ con $K^2 = 3e \Leftrightarrow S \simeq \mathbb{B}^2/\Gamma$

Teo (persson 1981) Dados $x, y \in \mathbb{Z}^+$ tales que

$$\frac{1}{5}(x-36) \leq y \leq 2x,\quad 2y \neq x-k,$$

con $k=2$, k impar $1 \leq k \leq 15$, o $k=19$

$\Rightarrow \exists S$ Sp. minimal de tipo general con $\pi_1(S) = \{1\}$,

$K^2_S = y$, $e(S) = x$

Cubrimientos cíclicos. ③

Y var projectiva suave $/\mathbb{C}$, $\dim(Y) = m$.

D divisor con cruces simples normales (i.e. $D \underset{loc}{\simeq} \underset{\text{en dim } 2}{\{xy=0\}}$)

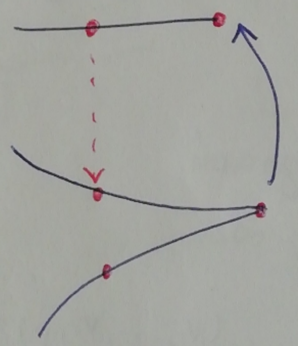
$\mathcal{L}$ fibrado ~~en rectas~~ lineal en Y tal que $\mathcal{L}^{\otimes n} \simeq \mathcal{O}_Y(D)$.

Data $\left(Y, D = \sum_{i=1}^{r} \nu_i D_i, n, \mathcal{L}\right)$

podemos construir: $X \xrightarrow[{[n:1]}]{f} Y$ que ramifica en D.

idea local

$\mathbb{A}^1$



$\mathbb{A}^2 \supseteq \{x^3 - y^2\}$

$$\begin{array}{c} \mathbb{C}[X] \\ \downarrow \\ \mathbb{C}[X,Y]/\langle X^3 - Y^2\rangle \end{array} \qquad , \qquad \frac{x^3 = y^2}{\text{raíz 2 de } x^3}$$

$\mathcal{L}^{-n} \simeq \mathcal{O}_Y(-D)$

$\rightsquigarrow 0 \to \mathcal{L}^{-n} \simeq \mathcal{O}_Y(-D) \to \mathcal{O}_Y \to \mathcal{O}_D \to 0$

Da estructura de $\mathcal{O}_Y$-alg a $\bigoplus_{i=0}^{n-1} \mathcal{L}^{-i}$

$\rightsquigarrow \operatorname{Spec}_{\mathcal{O}_Y}\left(\bigoplus_{i=0}^{n-1} \mathcal{L}^{-i}\right) = Y' \xrightarrow{f_1} Y.$

Normalización local.

C anillo local regular, x parámetro local, u unidad

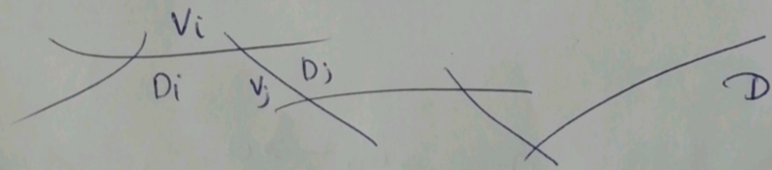
$\Longrightarrow B = C[t]/\langle t^n - x^v u\rangle$. La normalización A de B como C-mód está generada por $t^i x^{\left[\frac{vi}{n}\right]}$, $0 \leq i \leq n-1$.

- $\bar{Y} \xrightarrow{f_2} Y'$, f_2 normalización.

con $\bar{Y} = \mathrm{Spec}\left(\bigoplus_{i=0}^{n-1} \mathcal{L}^{-(i)}\right)$, $\mathcal{L}^{(i)} := \mathcal{L}^i \otimes \mathcal{O}_Y\left(-\sum_{j=1}^{r}\left[\frac{v_j \cdot i}{n}\right] D_j\right)$

- $X \xrightarrow{f_3} \bar{Y}$ desingularización.

- Caso Superficies $\quad n = p$, $(v_1, \dots, v_r, p) = 1$

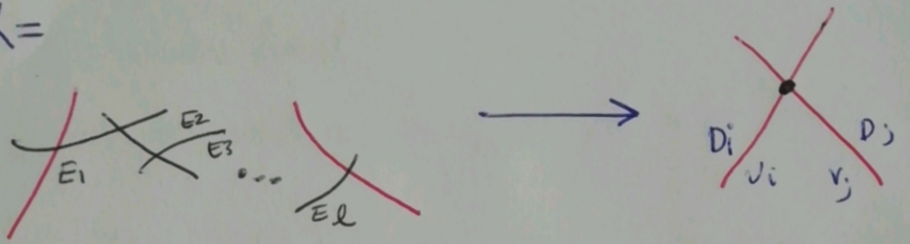


$$D = \sum_{i=1}^{r} v_i D_i \sim \mathcal{L}^p$$

$v_j q + v_i \equiv 0 \pmod p$, $\quad 0 < q \leq p-1$

$$\frac{p}{q} = e_1 - \cfrac{1}{e_2 - \cfrac{1}{\ddots \cfrac{1}{e_l}}} =: [e_1, \dots, e_l]$$

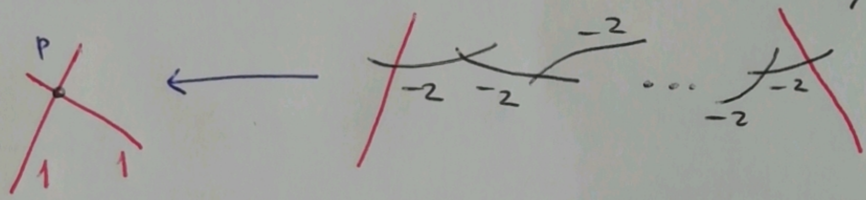
$X =$



$\subseteq Y$

$E_i \simeq \mathbb{P}^1, \quad E_i^2 = -e_i.$

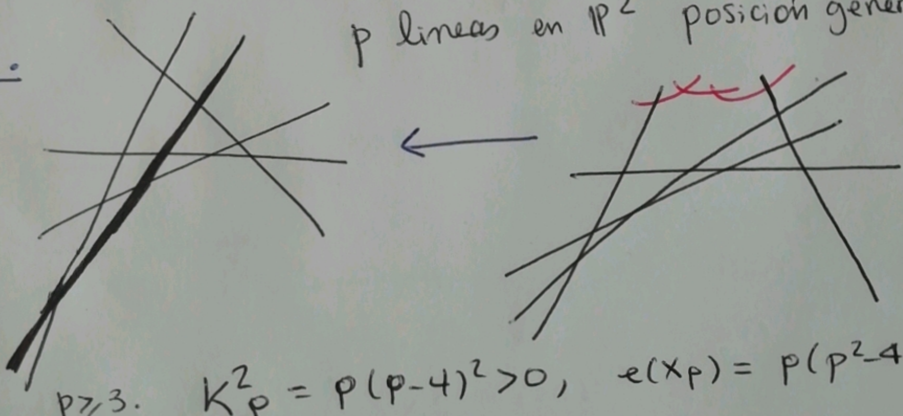
E.g.



$$\frac{p}{p-1} = \underbrace{[2,2,\ldots,2]}_{(p-1)-2's}$$

[ojo: X no canónico pues tiene curvas -2]

E.g. p lineas en $\mathbb{P}^2$ posición general.



$= X_p$

$p \geq 3. \quad K_p^2 = p(p-4)^2 > 0, \quad e(X_p) = p(p^2-4p+6) > 0$

$$\frac{1}{3} \leq \frac{K_p^2}{e(X_p)} \leq 1.$$