Goal: HMS for toric varieties

2 approaches: "B-side" "A-side"

$$D^b(X_\Sigma) \longrightarrow FS(T^*T^n, W_\Sigma) \longrightarrow \text{"} W_\Sigma^{-1}(+\infty) \text{"}$$

① $D^b(X_\Sigma) \xrightarrow{\text{Abouzaid, Hanlon-Hicks}} \mathrm{W\,Fuk}(T^*T^n, f_\Sigma)$

AHH: $\mathcal{O}(D) \longmapsto \mathcal{L}(D)$ "tropical Lagrangian sections"

expected to be $\simeq$

② $D^b(X_\Sigma) \xrightarrow{\text{FLTZ/K.}} Sh(T^n, f_\Sigma) \xrightarrow{\text{Ganatra-Pardon-Shende}} \mathrm{w\,Fuk}(T^*T^n, f_\Sigma)$

coherent constructible correspondence

KGPS: $\mathcal{O}(D) \longmapsto P(D) \longmapsto \mathcal{L}'(D)$

twisted polytope sheaf

Focus on this part. Today: B-side

Construction of (normal) toric varieties

Fix integral lattices

$$N \cong \mathbb{Z}^d, \quad M = N^\vee \cong \mathbb{Z}^d$$

$$N_{\mathbb{R}} := N \otimes \mathbb{R} \cong \mathbb{R}^d \qquad M_{\mathbb{R}} := M \otimes \mathbb{R} \cong \mathbb{R}^d$$

<u>Defn</u>: A <u>cone</u> in $N_{\mathbb{R}}$ is a strictly convex $\mathbb{R}_{\geq 0}$-linear combination of elements in N.

A <u>fan</u> is a finite set of cones Σ s.t.

- $\forall \sigma \in \Sigma$, $\tau \overset{\text{face}}{\subseteq} \sigma \Rightarrow \tau \in \Sigma$
- $\forall \sigma, \sigma' \in \Sigma \Rightarrow \sigma \cap \sigma'$ is a face of each

<u>Defn</u>: $\sigma \underset{\text{cone}}{\subseteq} N_{\mathbb{R}}$

$$\sigma^\vee = \{u \in M_{\mathbb{R}} : \langle u, v \rangle \geq 0 \ \forall v \in \sigma\}$$

dual cone

$\sigma^\vee \cap M$ monoid $\Rightarrow$ $\mathbb{C}[\sigma^\vee \cap M]$ monoid

$$z^m \cdot z^{m'} = z^{m+m'}$$

<u>Defn</u>: $\Sigma \subseteq N_{\mathbb{R}}$ fan

$$X_\Sigma := \operatorname*{colim}_{\sigma \in \Sigma} \operatorname{Spec} \mathbb{C}[\sigma^\vee \cap M]$$

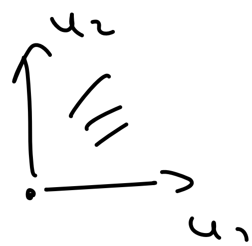
is a <u>toric variety</u>.

Example:

① $\Sigma = \{ \{0\} \}$

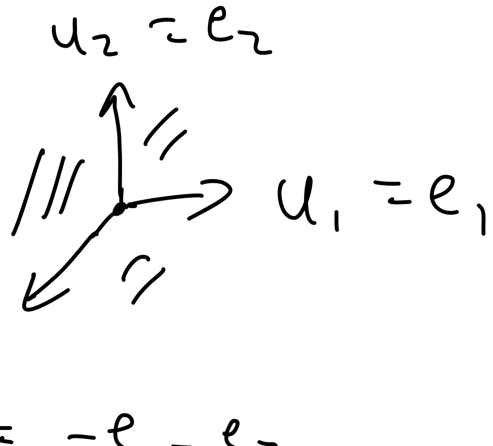
$$X_\Sigma = \operatorname{Spec} \mathbb{C}[x_1^{\pm}, \dots, x_d^{\pm}]$$

$$\cong T_N \cong N \otimes \mathbb{C}^\times$$

② $\Sigma =$ 

$\subseteq \mathbb{R}^2$

$$X_\Sigma = \operatorname{colim} \left[\begin{tikzcd} & (\mathbb{C}^\times)^2 \arrow[dl] \arrow[dr] & \\ (\mathbb{C}^\times) \times \mathbb{C} \arrow[dr] & & (\mathbb{C}^\times) \times \mathbb{C} \arrow[dl] \\ & \mathbb{C}^2 & \end{tikzcd} \right]$$

③ $\Sigma =$ 

$$X_\Sigma = \operatorname{colim}\left[\begin{tikzcd}[column sep=small] & & \mathbb{C}^\times \times \mathbb{C} \arrow[dll] \arrow[drr] & & \\ \mathbb{C}^2_{13} & & & & \mathbb{C}^2_{23} \\ & & (\mathbb{C}^\times)^2 \arrow[uu, "3"] \arrow[dl, "1"'] \arrow[dr, "2"] & & \\ & \mathbb{C}^\times \times \mathbb{C} \arrow[ul] \arrow[dr, hook] & & \mathbb{C}^\times \times \mathbb{C} \arrow[ur] \arrow[dl] & \\ & & \mathbb{C}^2_{12} & & \end{tikzcd}\right]$$

$$\cong \mathbb{P}^2$$

Orbit-Cone Correspondence

$\mathbb{T}_N = (\mathbb{C}^\times)^n \curvearrowright (\mathbb{C}^\times)^n$ naturally extends !

$X_\Sigma \supseteq$ orbit closures $\overset{1-1}{\longleftrightarrow}$ Σ

toric subvariety

codim k $\quad X_{p(\mathrm{star}(\sigma))}$, where $\longleftarrow\!\!\mapsto$ σ $\in$ dim k

$N_{\mathbb{R}} \xrightarrow{p} N_{\mathbb{R}} / N_{\mathbb{R},\sigma}$

$\forall\ \rho \in \Sigma(1) \longleftrightarrow \mathbb{T}$ - inv. divisor in X_Σ

$\sigma \in \Sigma(\text{top dim}) \longleftrightarrow \mathbb{T}$ - fixed pts

Divisors

$m \in M \Rightarrow \chi^m : \mathbb{T}_N \longrightarrow \mathbb{C}^\times$ rational function on X_Σ

$f \mapsto \mathrm{ord}_D(f)$

$\forall \rho \in \Sigma(1) \rightarrow$ DVR $\mathcal{O}_{X_\Sigma, D_\rho}$ with valuation $\nu_\rho : \mathbb{C}(X_\Sigma) \to \mathbb{Z}$

Lemma: $\nu_\rho(\chi^m) = \langle m, u_\rho \rangle$

Proof: extend u_ρ to a basis $\underset{e_1}{\overset{\shortparallel}{u_\rho}}, e_2, \dots, e_n$ of N

$\Rightarrow U_\rho = \mathrm{Spec}\, \mathbb{C}[x_1, x_2^{\pm}, \dots, x_n^{\pm}] = \mathbb{C} \times (\mathbb{C}^\times)^{n-1}$

$D_\rho \cap U_\rho = \{x_1 = 0\}$

$\Rightarrow \mathcal{O}_{X_\Sigma, D_\rho} = \mathcal{O}_{U_\rho, U_\rho \cap D_\rho} = \mathbb{C}[x_1, \dots, x_n]_{\langle x_1 \rangle}$

$f \in \mathbb{C}(x_1, \dots, x_n)^*$ $\quad \nu_\rho(f) = l \in \mathbb{Z}$

if $f = x_1^l \frac{g}{h}$, $g, h \in \mathbb{C}[x_1, \dots, x_n] \setminus \langle x_1 \rangle$

$f = \chi^m$

$\Rightarrow \chi^m = x_1^{\langle m, u_\rho \rangle} x_2^{\langle m, e_2 \rangle} \cdots x_n^{\langle m, e_n \rangle}$ $\quad \boxtimes$

Cor: $\mathrm{div}(\chi^m) = \sum_{\rho \in \Sigma(1)} \langle m, u_\rho \rangle D_\rho$, $\quad u_\rho$ primitive ray generator

Pf: $\dim(\chi^m) = \sum_{\rho \in \Sigma(1)} \nu_{D_\rho}(\chi^m) D_\rho$

Thm: Suppose X_Σ has no torus factor i.e. $\{u_\rho\}_{\rho \in \Sigma(1)}$ spans $N_{\mathbb{R}}$

Have short exact sequence

$$0 \to M \xrightarrow{i} \mathrm{Div}_{T_N}(X_\Sigma) \xrightarrow{\mu} \mathrm{Cl}(X_\Sigma) \to 0$$

smooth $/\!/$ $\cup\!|$ finit index

$\mathrm{Pic}(X_\Sigma)$ $\mathrm{Pic}(X_\Sigma)$ simplicia

$$m \longmapsto \sum_{\rho \in \Sigma(1)} \langle m, u_\rho \rangle D_\rho$$

$$\sum_{\rho \in \Sigma(1)} a_\rho D_\rho \longmapsto \sum_{\rho \in \Sigma(1)} a_\rho [D_\rho]$$

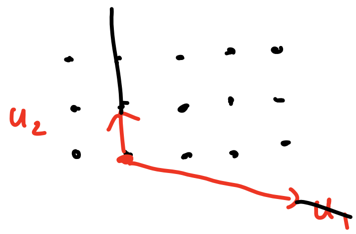
Proposition: $D = \sum a_\rho D_\rho$ Cartier

$\iff \exists$ local data $\{U_\sigma, m_\sigma\}_{\sigma \in \Sigma}$, $m_\sigma \in M$ s.t.

$$\langle m_\sigma, u_\rho \rangle = -a_\rho \quad \forall \rho \in \Sigma(1)$$

$$\iff \sum_{\rho \in \sigma(1)} a_\rho D_\rho = D|_{U_\sigma} = \mathrm{div}(\chi^{-m_\sigma})|_{U_\sigma}$$

Ex: $\sigma = \mathrm{Cone}(\underset{\substack{\parallel \\ u_1}}{de_1 - e_2}, \underset{\substack{\parallel \\ u_2}}{e_2}) \subseteq \mathbb{Z}^2$



$$\Rightarrow U_\sigma = \mathbb{C}^2 / \underbrace{\mathbb{Z}/d\mathbb{Z}}_{\text{act by } \begin{pmatrix} d & -1 \\ 0 & 1 \end{pmatrix}}$$

$$\begin{array}{l} Cl(U_\sigma) = \mathbb{Z}_{\rho_1} \oplus \mathbb{Z}_{\rho_2} / \sim \\ \quad \parallel \\ \mathbb{Z}/d\mathbb{Z} \end{array}$$

$$0 \sim \operatorname{div}(x^{e_1}) = \langle e_1, u_1 \rangle D_1 + \langle e_2, u_2 \rangle D_2$$
$$= d D_1$$
$$0 \sim \operatorname{div}(x^{e_2}) = \langle e_2, u_1 \rangle D_1 + \langle e_2, u_2 \rangle D_2$$
$$= -D_1 + D_2$$

<u>Support fn</u>: $\Sigma \subseteq N_{\mathbb{R}}$

A support fn is a fn $\varphi : |\Sigma| \to \mathbb{R}$ linear on each cone

$$SF(\Sigma) = \{ \text{support fns on } \Sigma \}$$

<u>Thm</u>: Given D cartier the function

$$\varphi_D : |\Sigma| \to \mathbb{R}$$
$$u \mapsto \varphi_D(u) = \langle m_\sigma, u \rangle, \quad u \in \sigma$$

is a well-defined support function

$$\Rightarrow \quad CDiv_{T_N}(X_\Sigma) \xrightarrow{\sim} SF(\Sigma, N)$$
$$D \mapsto \varphi_D, \qquad \varphi_D(u_\rho) = -a_\rho$$

Divisor Polytope: $\quad D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho \quad$ nef

$$\emptyset \neq P_D := \{ m \in M_{\mathbb{R}} : \langle m, u_\rho \rangle \geq -a_\rho \}$$

$$\underset{\text{lattice polytope}}{\subseteq} M_{\mathbb{R}}$$

geometry of X_Σ can be read off from Σ

E.g.

Σ	X_Σ
$\lvert\Sigma\rvert = N_{\mathbb{R}}$	proper
$\{u_\rho\}_{\rho \in \sigma(1)}$ is a $\mathbb{Z}$-basis for each $\sigma \in \Sigma$	smooth
$\mathbb{R}$-basis	simplicial
$\exists$ strictly convex support fn φ_D for D ample	projective

Σ = normal fan of P_D

Computing Cohomology of $\mathcal{O}(D)$

Given $D = \sum_\rho a_\rho D_\rho$, $m \in M$, define

- $V_{D,m} = \bigcup_{\sigma \in \Sigma} \operatorname{Conv}(u_\rho \mid \rho \in \sigma(1), \langle m, u_\rho \rangle < -a_\rho)$

- $V_{D,m}^{supp} = \{ u \in |\Sigma| : \langle m, u \rangle < \varphi_D(u) \}$

<u>Thm</u>: ① $H^p(X_\Sigma, \mathcal{O}_{X_\Sigma}(D))_m \simeq \tilde{H}^{p-1}(V_{D,m}, \mathbb{C})$

② $D = \mathbb{Q}$ - Cartier $\Rightarrow$ $H^p(X_\Sigma, \mathcal{O}_{X_\Sigma}(D))_m \cong \tilde{H}^{p-1}(V_{D,m}^{supp}, \mathbb{C})$

<u>Corollaries</u>:

1) <u>Demazure Vanishing</u>:

$|\Sigma|$ convex , D nef $\Rightarrow$ φ_D convex

$H^p(X_\Sigma, \mathcal{O}_{X_\Sigma}(D)) = 0 \quad \forall\, p > 0$

2) $H^0(X_\Sigma, \mathcal{O}_{X_\Sigma}(D)) = \mathbb{C}\langle P_D \cap M \rangle$.

<u>Connection to $D^b(X_\Sigma)$</u>:

$\mathcal{O}(D)$, $D \in Cl(X_\Sigma)$ <u>generate</u> $D^b(X_\Sigma)$.

$R\operatorname{Hom}(\bigoplus_{D \in Cl(X_\Sigma)} \mathcal{O}(D), \mathcal{F}) = 0 \Rightarrow \mathcal{F} = 0$

Hence, enough to understand $D^b(X_\Sigma)$ by looking at $\mathcal{O}(D)$ and their RHom.