<u>Last time</u> : Fundamental SES for toric variety X_Σ

$$0 \to M \xrightarrow{i} \mathrm{Div}_{T_N}(X_\Sigma) \xrightarrow{\mu} \mathrm{Cl}(X_\Sigma) \to 0$$

$$m \longmapsto \sum_{\rho \in \Sigma(1)} \langle m, u_\rho \rangle D_\rho$$

$$\sum a_\rho D_\rho \longmapsto \sum a_\rho [D_\rho]$$

<u>Support fn</u> : $\Sigma \subseteq N_{\mathbb{R}}$

A support fn is a fn $\varphi : |\Sigma| \to \mathbb{R}$ linear on each cone

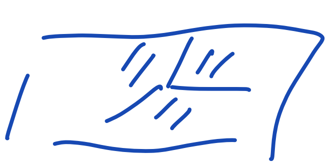
$$SF(\Sigma) = \{ \text{support fns on } \Sigma \}$$

<u>Thm</u>: Given D cartier the function

$$\varphi_D : |\Sigma| \to \mathbb{R}$$
$$u \longmapsto \varphi_D(u) = \langle m_\sigma, u \rangle, \quad u \in \sigma$$

is a well-defined support function

$$\Rightarrow \quad \mathrm{CDiv}_{T_N}(X_\Sigma) \xrightarrow{\sim} SF(\Sigma, N)$$
$$D \longmapsto \varphi_D, \quad \varphi_D(u_\rho) = -a_\rho$$

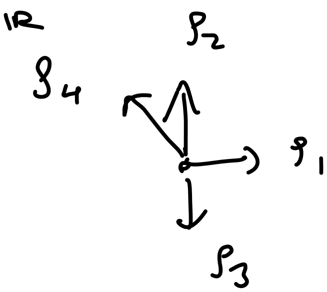


Divisor Polytope: Given $D = \sum_{\rho\in\Sigma(1)} a_\rho D_\rho$

$$P_D := \{ m \in M_{\mathbb{R}} : \langle m, u_\rho \rangle \geq -a_\rho \} \subseteq M_{\mathbb{R}}$$

lattice polytope

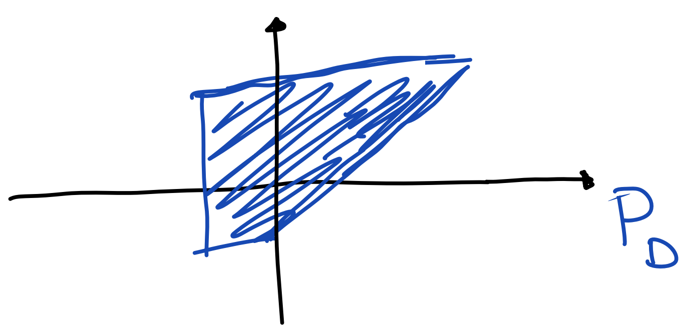
- D nef $\Rightarrow$ $P_D \neq \emptyset$
- D ample $\Rightarrow$ $\dim P_D = \dim M_{\mathbb{R}}$



Ex: $Bl_0 \mathbb{P}^2 = \mathbb{F}_1$

$$D = D_1 + D_2 + 2D_3 + D_4$$

$$\Rightarrow P_D = \{ m = (x,y) \in \mathbb{R}^2 : x \geq -1, \; 2 \geq y \geq -1, \; -x+y \geq -1 \}$$



Computing Sheaf Cohomology

Recall: $X_\Sigma = \operatorname{colim}_{\sigma \in \Sigma} X_\sigma$ affine

$\Rightarrow$ standard Čech cover $\{X_\sigma\}_{\sigma \in \Sigma_{max}} = \mathcal{U}_\Sigma$

$\Rightarrow$ Čech cpx $\check{C}(\mathcal{U}_\Sigma, \tilde{F})$:

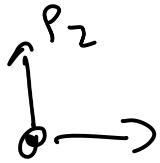
$$0 \to \bigoplus_{\sigma} \Gamma(U_\sigma, \tilde{F}) \longrightarrow \bigoplus_{\sigma_1,\sigma_2} \Gamma(U_{\sigma_1} \cap U_{\sigma_2}, \tilde{F})$$

$$\to \cdots \xrightarrow{d = \sum \pm \text{res}} \bigoplus_{\sigma_0,\dots,\sigma_p} \Gamma(U_{\sigma_0} \cap \cdots \cap U_{\sigma_p}, \tilde{F}) \to \cdots$$

$$\Rightarrow \check{H}^p(\mathcal{U}_\Sigma, \tilde{F}) = H^p(X_\Sigma, \tilde{F})$$

Rmk: character decomp

$$H^p(X_\Sigma, \tilde{F}) = \bigoplus_{m \in M} H^p(X_\Sigma, \tilde{F})_m$$

Ex: $\Sigma =$ 

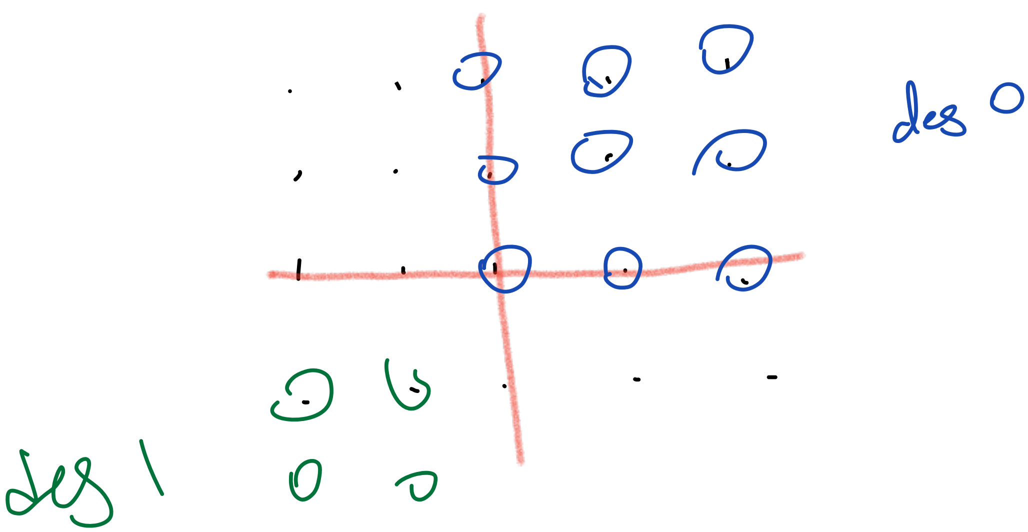
(no 2-cones)

$$\Rightarrow X_\Sigma = \mathbb{A}^2 \setminus \{0\}$$

$H^0(X_\Sigma, \mathcal{O}) = ?$.

$$\underbrace{H^0(U_1, \mathcal{O})}_{\mathbb{C}[x, y^{\pm 1}]} \oplus \underbrace{H^0(U_2, \mathcal{O})}_{\mathbb{C}[x^{\pm 1}, y]} \xrightarrow{f_1 - f_2} \underbrace{H^0(U_1 \cap U_2, \mathcal{O})}_{\mathbb{C}[x^{\pm 1}, y^{\pm 1}]}$$

$\Rightarrow \quad H^0 = \mathbb{C}[x,y] \qquad H^1 = \bigoplus_{\substack{i<0 \\ j<0}} \mathbb{C}\cdot x^i y^j$



Computing Cohomology of $\mathcal{O}(D)$ using φ_D

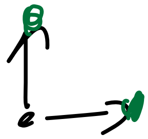
Given $D = \sum_\rho a_\rho D_\rho$, $m \in M$,

- $V_{D,m} = \bigcup_{\sigma \in \Sigma} \mathrm{Conv}(u_\rho \mid \rho \in \sigma(1),\ \langle m, u_\rho \rangle < -a_\rho)$
- $V^{supp}_{D,m} = \{ u \in |\Sigma| : \langle m, u \rangle < \varphi_D(u) \}$

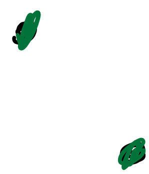
$\subseteq N_{\mathbb{R}}$

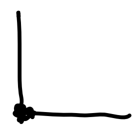
Thm: ① $H^p(X_\Sigma, \mathcal{O}_{X_\Sigma}(D))_m \simeq \tilde{H}^{p-1}(V_{D,m}, \mathbb{C})$

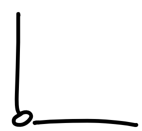
② D = $\mathbb{Q}$-Cartier $\Rightarrow H^p(X_\Sigma, \mathcal{O}_{X_\Sigma}(D))_m \cong \tilde{H}^{p-1}(V^{supp}_{D,m}, \mathbb{C})$

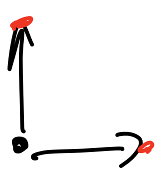
Eg: (1)  $D = 0, \quad m = (-1,-1)$

$\varphi_D = 0$

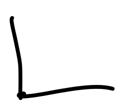
$\Rightarrow V_{D,m} =$ 

$\Rightarrow V^{supp}_{D,m} = \{(x,y) \in$  $: -x-y<0\}$

$=$  $\Rightarrow \tilde{H}^{p-1} = \begin{cases} \mathbb{C} & p=1 \\ 0 & \text{else} \end{cases}$

(2)  $D = 0, \quad m = (1,1)$

$\Rightarrow V_{D,m} = \phi$

$\Rightarrow V^{supp}_{D,m} = \{(x,y) \in$  $: x+y<0\} = \phi$

$\Rightarrow \tilde{H}^{p-1} = \begin{cases} \mathbb{C} & p=0 \\ 0 & \text{else} \end{cases}$

Corollaries :

1) Demazure Vanishing :

$$|\Sigma| \text{ convex}, \quad D \text{ nef} \Rightarrow \varphi_D \text{ convex}$$

$$H^p(X_\Sigma, \mathcal{O}_{X_\Sigma}(D)) = 0 \quad \forall\, p > 0$$

pf : Fix any $m \in M$ & let $u, v \in V_{D,m}^{supp}$ i.e.

$\langle m, u \rangle < \varphi_D(u)$, $\langle m, v \rangle < \varphi_D(v)$. Then

$0 \le t \le 1$ we have

$$\langle m, (1-t)u + tv \rangle = (1-t)\langle m, u \rangle + t\langle m, v \rangle$$
$$< (1-t)\varphi_D(u) + t\,\varphi_D(v)$$
$$\le \varphi_D((1-t)u + tv)$$

$\Rightarrow V_{D,m}$ is convex $\Rightarrow$ contractible

section polytope for $\mathcal{O}_L$

2) $H^0(X_\Sigma, \mathcal{O}_{X_\Sigma}(D)) = \mathbb{C}\langle P_D \cap M \rangle$.

Reason : $\forall\, m \in P_D = \{ m \in M_{\mathbb{R}} : \langle m, u_\rho \rangle \ge -a_\rho, \forall \rho \}$

$\Rightarrow V_{D,m} = \phi$ by defn.

Thm : (Batyrev - Borisov vanishing)

D anti-nef on X_Σ complete fan

Then $H^p(X_\Sigma, \mathcal{O}(D)) = \begin{cases} 0 & p \neq \dim P_D \\ \mathbb{C}\cdot\langle -P_{-D} \cap M \rangle & p = \dim P_D \end{cases}$

<u>Different perspective</u>: from const. shves on $M_{\mathbb{R}}$

stalk of const sheaf

- D nef $\Rightarrow H^\bullet(X_\Sigma, \mathcal{O}(D)) \cong \bigoplus_{m \in M} (\underline{\mathbb{C}}_{P_D})_m$
- D antinef $\Rightarrow H^\bullet(X_\Sigma, \mathcal{O}(D)) \cong \bigoplus_{m \in M} (\underline{\mathbb{C}}_{-P_D^\circ})_m$
- In $\mathbb{A}^2 \setminus \{0\}$ as we saw,

$$H^\bullet(X_\Sigma, \mathcal{O}(D)) \cong \bigoplus_{m \in M} \left(\begin{array}{c|c} 0 & \mathbb{C} \\ \hline \mathbb{C}[\,] & 0 \end{array} \right)_m$$

$$\mathcal{H}^\bullet\left(\left[\begin{array}{c} \underline{\mathbb{C}}_{\{x>0\}} \\ \oplus \\ \underline{\mathbb{C}}_{\{y>0\}} \end{array} \longrightarrow \underline{\mathbb{C}}_{\mathbb{R}^2} \right] \right)$$

We can already see a naive form of CCC

$$H^\bullet \text{ of } \mathcal{O}(D) \longleftrightarrow \bigoplus_{m \in M_{\mathbb{R}}} (\mathcal{P}_D)_m$$

"Čech-like" complex of polyhedral constructible shf