Cohomology of line bundles.

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Aim: Description of in terms of convex geometry of Δ.

Idea: Let $X = X(\Delta)$. Use the Cech covering $X(\Delta) = \bigcup_{\sigma \in \Delta} U_\sigma$ to compute $H^p(X, \mathcal{O}_X(\lfloor D \rfloor))$ for $D =$ $\mathbb{Q}$-T-divisor.

Suppose $D = \sum a_i D_i$, where the D_i are the T-invariant divs. corresponding to the ~~prim~~ prim rays v_i. Then,

$$\Gamma = (U_\sigma, \mathcal{O}(\lfloor D \rfloor)) = \bigoplus_u k\,\chi^u$$

where the sum is over $u \in M$ st. $\langle u, v_i \rangle + a_i \geq 0 \;\; \forall v_i \in \sigma$.

The pieces of $\Gamma(U_\sigma, \mathcal{O}\lfloor D \rfloor)_u$ can be put together into a simplicial complex.

The function of a $\mathbb{Q}$-T-divisor.

→ Let $D = \sum a_i D_i$ as above. Define $\psi_D : |\Delta| \to \mathbb{R}$ as follows:
If $v \in \sigma$, $\sigma = \langle v_1, \dots, v_r \rangle$, then $\psi_D(v) = \sup\{-\sum a_i \lambda_i \,/\, v = \sum_{i=1}^r \lambda_i v_i, \lambda_i \geq 0\}$ (*)

obs: If $v \in \tau \leq \sigma$, then in any expression $v = \sum \lambda v_i$, $\lambda_i = 0$ if $v_i \notin \tau$ $\Rightarrow$ ψ_D is well defined.

Lemma 1: The sup (*) is a maximum so finite. If $v \in N_\mathbb{Q} \cap |\Delta|$ then $\psi_D(v) \in \mathbb{Q}$. For all $u \in P_D := \{u \in M_\mathbb{R} \,/\, \langle u, v_i \rangle + a_i \geq 0 \; \forall i\}$,

$$\langle u, v \rangle \geq \psi_D(v).$$

Pf.: Let $\sigma = \langle v_1, \dots, v_r \rangle$, $v = \sum_{i=1}^r \lambda_i v_i$, $\lambda_i \geq 0$
Then $\forall u \in P_D$,

$$\langle u, v \rangle = \sum_{i=1}^r \lambda_i \langle u, v_i \rangle \geq -\sum_{i=1}^r \lambda_i a_i \Rightarrow \langle u, v \rangle \geq \psi_D(v)$$

In general, P_D might be ϕ, but not so if $X = U_\sigma$. Restrict to the case (as ψ_D is "locally defined") get that $\psi_D(v)$ is finite.

For the fact that (*) is max, and rational when $v \in N_{\mathbb{Q}} \cap |\Delta|$, see following example.

Ex: suppose $\sigma = \langle v_1, v_2, v_3, v_4 \rangle$ $v_1 + v_3 = v_2 + v_4$ with v_1, v_2, v_3 are lin. indep suppose $v = c_1 v_1 + c_2 v_2 + c_3 v_3$ and consider an expression $v = \sum_{i=1}^{4} \lambda_i v_i$, $\lambda_i \geq 0$.

$$\Rightarrow \quad \begin{aligned} \lambda_1 + \lambda_4 &= c_1 \\ \lambda_2 - \lambda_4 &= c_2 \\ \lambda_3 + \lambda_4 &= c_3 \end{aligned} \qquad \text{and } \sum a_i \lambda_i = \sum a_i c_i + \lambda_4 (a_2 - \ldots$$

We are maximizing a linear funct. in a closed interval. rationality is clear.

Exerc 1: Show that if D, E are $\mathbb{Q}$-T-div $\Rightarrow \psi_{D+E}(v) \leq \psi_D(v) + \psi_E(v)$ $\forall v \in |\Delta|$, with equality if D or E is a $\mathbb{Q}$-Cartier divisor.

Obs: If D is $\mathbb{Q}$-Cartier, then ψ_D agrees with the old definition.

→ Let D be $\mathbb{Q}$-T-divisor, $u \in M$. Consider $Z_D(u) := \{ v \in |\Delta| \,/\, \langle u, v \rangle \geq \psi_D(v) \}$ This is a closed nonempty cone.

Exerc: $P_D = \{ u \in M_{\mathbb{R}} \,/\, \langle u, v \rangle \geq \psi_D(v) \quad \forall v \in |\Delta| \}$

Thus, $u \in P_D \iff Z_D(u) = |\Delta|$.

Lemma 2: Let D be a $\mathbb{Q}$-T-div, $\sigma \in \Delta$. Then $\psi_D|_\sigma$ is concave. In particular, $\forall u \in M$, $\sigma \setminus Z_D(u)$ is convex.

Cohomology

Prop: If D is a $\mathbb{Q}$-T-div on $X = X(\Delta)$, then $\forall p$,

$$H^p(X, \mathcal{O}(\lfloor D \rfloor)) = \bigoplus_{u \in M} \underbrace{H^p(X, \mathcal{O}(\lfloor D \rfloor))_u}$$

pieces because of torus action.

and canonical isoms. $\forall u \in M$,

$$H^p(X, \mathcal{O}(\lfloor D \rfloor))_u = H^p(|\Delta|, |\Delta| - Z_D(u); k) \text{ (Topological)}.$$

Note: $H^p(|\Delta|, |\Delta| \setminus Z_D(u); k) =: H^p_{Z_D(u)}(|\Delta|; k)$.

obs: $H^0(X, \mathcal{O}(\lfloor D \rfloor)) = \bigoplus_{u \in P_D} k\chi^u$

Now, $u \in P_D \iff Z_D(u) = |\Delta| \iff H^0(|\Delta|, |\Delta| \setminus Z_D(u)) \neq 0$.
The prop. is a generalization of this.

→ Intermediate step. Given $D = \sum a_i D_i$, $u \in M$ define the following simplicial cx $\Sigma_D(u)$ on Δ. The set of cones $\{\sigma_0, \dots, \sigma_r\} \in \Sigma_D(u)$ $\iff \sigma_0 \cap \dots \cap \sigma_r \notin Z_D(u)$. Equivalently

Equivalently, $\{\sigma_0, \dots, \sigma_r\} \notin \Sigma_D(u) \iff \forall v_i \in \sigma_0 \cap \sigma_1 \cap \dots \cap \sigma_r$
$\langle u, v_i \rangle + a_i \geq 0$.

Thus, $\{\sigma_0, \dots, \sigma_r\} \notin \Sigma_D(u) \iff$ coeffs D_i in $D + \mathrm{div}(\chi^u) \geq 0$.

Denote by Σ the full simplicial complex of Δ.

Lemma 3. If D is a $\mathbb{Q}$-T-div on $X(\Delta)$, then $\forall p$, $H^p(X, \mathcal{O}(\lfloor D \rfloor)) = \bigoplus_{u \in M} H^p(X, \mathcal{O}(\lfloor D \rfloor))_u$

donde $H^p(X, \mathcal{O}_X(\lfloor D \rfloor))_u \simeq H^p(\Sigma, \Sigma_D(u); k)$

Pf. Use Čech covering $\bigcup_{\sigma \in \Delta} U_\sigma$, $H^0(U_\sigma, \mathcal{O}_X(\lfloor D \rfloor)) = \bigoplus_{\substack{u : \langle u, v_i \rangle + a_i \geq 0 \\ v_i \in \sigma}} k\chi^u$

Note $\langle u, v_i \rangle + a_i \geq 0 \; \forall v_i \in \sigma \iff \{\sigma\} \notin \Sigma_D(u)$

The Čech differentials preserve the grading, so $H^p(X, \mathcal{O}(\lfloor D \rfloor)) = \bigoplus_{u \in M} H^p(X, \mathcal{O}(\lfloor D \rfloor))_u$

The Čech cochains in degree u can be identified with the complex of simplicial cochain for the pair $(\Sigma, \Sigma_D(u))$

Thus we have a canonical, $H^p(X, \mathcal{O}(\lfloor D \rfloor))_u \simeq H^p(\Sigma, \Sigma_D(u); k)$.

Obs. If $|\Delta|$ is convex, and ψ_D is concave $\Rightarrow H^p(X, \mathcal{O}(\lfloor D \rfloor)) = 0 \; \forall p \geq 1$
Since $|\Delta|$ is always contractible, the long exact seq in cohomol.

$\Rightarrow H^p(|\Delta|, |\Delta| \setminus Z_D(u)) \xrightarrow{\sim} \check{H}^{p-1}(|\Delta| \setminus Z_D(u))$ (difference in degree 0)

convex because ψ_D is concave.