Resolución de singularidades tóricas

David Grim.

Recall : a fan Δ' is called refinement of a fan Δ if every $\sigma \in \Delta$ is a union of cones in Δ'

Given X = variety, a ~~st~~ resolution of sing. is a proper birat. morphism
$Y \to X$
smooth $\cup$ ~~...~~ The existence in char$(k) = 0$ (Hironaka)
Excep. fiber $\to$ Sing(X)

Thm : $k = \bar{k}$ any field (of any char) and X = toric variety, there exists a smooth toric variety Y together with a proper birational toric morphism $f : Y \to X$.

IDEA : Given $X(\Delta)$ find singular refinement Δ' of Δ such that $Y = X(\Delta') \to X(\Delta)$ will be the resolution.

(birational is clear since same torus)

$\to$ Study "resolution / good refinements" of fans.

Def : For $\sigma \subset \mathbb{R}^n$ rational polyhedral simplicial cone. Let $v_1, \dots, v_r \in \mathbb{Z}$ be the primitive elements of its rays, we define

$$\operatorname{mult}(\sigma) := \#\left(\langle \sigma \cap \mathbb{Z}^n \rangle_{\mathbb{Z}} / \langle v_1, \dots, v_r \rangle_{\mathbb{Z}} \right)$$

pts inside convex sum

Note σ is nonsingular $\iff \operatorname{mult}(\sigma) = 1$.

$$\left(\text{def } \operatorname{mult}(\sigma) := \frac{\operatorname{Vol}(v_1, \dots, v_r)}{\operatorname{Vol}(e_1, \dots, e_n)} \right)$$
chosen correctly

Def : Let $\Delta \subset \mathbb{R}^n$ rat. polyh. fan and $v \in |\Delta| \cap \mathbb{Z}^n \setminus \{0\}$
$\Rightarrow$ Define star-subdivision $\Delta(v)$ is the fan consisting of the following for every $\sigma \in \Delta$ with $v \notin \sigma$ the cones σ (we keep them) and, if σ is a face of some τ with $v \in \tau$, we also take $\sigma(v) := \operatorname{pos}(\sigma, v)$.

$$\operatorname{pos}(\sigma, v) = \langle \sigma, v \rangle_{\mathbb{R}_+}$$

Prop: $\Delta(v)$ is a refinement of Δ.

Remark: Using iterated star-subdivisions we obtain a simplicial fan.

Lemma ★: Let Δ be a simplicial fan, $\sigma \in \Delta$ cone.
Let $v \in \mathbb{Z}^n \cap \underset{\text{interior}}{\sigma^\circ}$. Denote $m := \max\{\text{mult}(\sigma') \,/\, v \in \sigma'\}$
$v = \sum t_i v_i$, with $0 \leq t_i < 1$
$\Rightarrow$ for every $\tau(v) \in \Delta(v) \setminus \Delta$, we have $\text{mult}(\tau(v)) < m$.

Cor: toric resolution of singularities of $X = X(\Delta)$, $\Delta \subseteq \mathbb{R}^n$.
$m := \max\{\text{mult}(\sigma) \,/\, \sigma \in \Delta\}$ $\quad k := \#\{\sigma \in \Delta \,/\, \text{mult}(\sigma) = m\}$

with the lemma: can repeat ~~inter~~ to make mult smaller ... ■

Lemma 1: Let σ be a simplicial cone, τ a face of σ
$\Rightarrow$ $\text{mult}(\tau) \leq \text{mult}(\sigma)$.

Proof of Lemma ★: $\tau(v) = \langle \tau, v \rangle_{\mathbb{R}_+}$, τ was a face of $\tilde{\sigma} \in \Delta$ containing v

The prim. ray generators $v_1, .., v_r$ of σ extend to generators of ... because of determinant properties (see Mircea's notes) ■ $\quad \tau < \tilde{\sigma} < \sigma$, $\tilde{\sigma} \ni v$

See toric surfaces.