# Derived Categories & S.O.D's

$X$ = smooth, proj, $k = \mathbb{C}$

eventually zero

<u>Def</u>: $D^b(X) = \{ 0 \to \underset{\substack{\cap \\ \operatorname{Coh} X}}{F_i} \to F_{i+1} \to \cdots 0 \}$

+ quasi-isomorphisms are invertible

e.g. $0 \to A \to B \to C \to 0$ exact in $\operatorname{Coh} X$

$$\Rightarrow \quad A \simeq [\overset{(0)}{B} \to \overset{(1)}{C}] \quad \text{in } D^b(X)$$

$$C \simeq [\underset{(-1)}{A} \to \underset{(0)}{B}]$$

<u>Rmk</u>

- $\operatorname{Coh} X$ = Abelian Category (has short exact seq's)
- $D^b(X)$ = <u>Triangulated</u> Category (has <u>exact triangles</u>)

If $0 \to A \to B \to C \to 0$ exact in $\operatorname{Coh}(X)$

$$\Rightarrow \quad A \to B \to C \to A[1] \quad \text{exact triangle in } D^b(X)$$

shift the complex to the left

Sometimes write as:

$$\begin{tikzcd} A \arrow[r] & B \arrow[dl] \\ C \arrow[u, dashed] & \end{tikzcd}$$

Remark

(1) $D^b(X)$ is the natural environment for sheaf cohomology & other derived functors

e.g. $\Gamma : \mathrm{Coh}\, X \longrightarrow \mathrm{Vect}_k$ functor of global sections

It is left exact

⇝ Define its right derived functor, which is

$$R\Gamma : D^b(X) \longrightarrow D^b(\mathrm{Vect}_k)$$

which is to apply $\Gamma$ to a resolution,

say $\underset{\mathrm{Coh}\, X}{\mathcal{F}} \simeq [ I^0 \longrightarrow I^1 \longrightarrow \cdots ]$

quasi-isom.

injectives

& $H^i := R^i\Gamma : D^b(X) \longrightarrow \mathrm{Vect}_k$

is the i-th derived functor

i.e. take i-th cohomology of the resulting chain cpx in $D^b(\mathrm{Vect}_k)$

Similarly:

e.g. $- \otimes F : \mathrm{Coh}\, X \longrightarrow \mathrm{Coh}\, X$

It is right exact

⇝ Define its left derived functor

by taking left resolutions by vector bundles

⚠ (injectives are not coherent, and locally free bdls are not projective, but one can show that these derived functors can be defined and work fine...)

(2) $D^b(X)$ is studied in Mirror Symmetry
ie $D^b(X)$ should be = Fukaya Cat. of some $Y$
where $Y$ is a symplectic manifold

(3) Bridgeland Stability conditions on $D^b(X)$
allow us to study certain interesting moduli spaces
(e.g. say bundles over a K3) whose entire
birational geometry is encoded in variation of
the stability conditions.

(4) $D^b(X)$ is supposed to tell us something
about birational geometry of $X$.
There are conjectures of rationality
& possible birational invariants.

We will say more about (4).

<u>First</u>: to study $D^b(X)$ we want to "break it into pieces", so:

<u>Def:</u>

$D^b(X) = \langle \mathcal{A}_1, \ldots, \mathcal{A}_m \rangle$ is a Semi-Orthogonal Decomposition $= S.O.D.$

if

i) Each $\mathcal{A}_j \subset D^b(X)$ is an "admissible" subcategory

i.e. the inclusion $\mathcal{A}_j \overset{i}{\hookrightarrow} D^b(X)$ has left & right adjoints

ii) There is no morphisms from right to left:

$$\mathrm{Hom}_{D^b(X)}(\mathcal{A}_j, \mathcal{A}_k) = 0 \qquad j > k$$

( basically means RHom, i.e. Ext )

iii) They generate $D^b(X)$:

i.e. $D^b(X)$ is the smallest <u>triangulated</u> subcategory containing all $\mathcal{A}_1, \ldots, \mathcal{A}_m$

<u>Obs:</u>

(iii) means every $\mathcal{F} \in D^b(X)$ is an iterated extension.

$$\begin{tikzcd}
0 = \mathcal{F}_m \arrow[r] & \mathcal{F}_{m-1} \arrow[dl] & \cdots & \mathcal{F}_2 \arrow[r] & \mathcal{F}_1 \arrow[r] \arrow[dl] & \mathcal{F} \arrow[dl] \\
A_m \arrow[u, dashed] & & & A_2 \arrow[u, dashed] & A_1 \arrow[u, dashed] &
\end{tikzcd}$$

with $A_j \in \mathcal{A}_j$

e.g. say $D^b(X) = \langle \mathcal{A}, \mathcal{B} \rangle$ is S.O.D.

(i) says $\mathcal{A} \underset{a}{\overset{i}{\rightleftarrows}} D^b(X)$

has (left) adjoint $a$

Let $F \in D^b(X)$

By (iii) $\exists$ $A \in \mathcal{A}$, $B \in \mathcal{B}$

$$\text{s.t.} \quad B \to F \to A \to B[1] \qquad (*)$$

Here can put $A = a(F)$

Similar for $B$.

In general, if $\mathcal{A} \subset D^b(X)$ is admissible, can write $D^b(X) = \langle \mathcal{A}, {}^{\perp}\mathcal{A} \rangle$

where ${}^{\perp}\mathcal{A} := \{ \mathrm{Hom}(-, \mathcal{A}) = 0 \}$.

. Observe in $(*)$, $B \to F \to A$ is indeed zero from (ii).

$$\underbrace{B \to F \to A}_{0}$$

<u>Remark</u>

- $E$ is exceptional $\iff$ $\langle E \rangle$ is admissible
  i.e. $R\mathrm{Hom}(E) = \mathbb{C}$

- $E_1, \ldots, E_m$ is exceptional collection
  $\iff D^b(X) = \langle E_1, \ldots, E_m, \mathcal{B} \rangle$ is S.O.D.

  where $\mathcal{B} :=$ (left) orthogonal to $\langle E_1, \ldots, E_m \rangle$

- $E_1, \dots, E_m$ is <u>full</u> $\iff \mathcal{B} = 0$.

## Examples

1) $D^b(\mathbb{P}^n) = \langle \mathcal{O}(-n), \dots, \mathcal{O} \rangle$

2) $D^b(X)$ is indecomposable if $K_X = 0$.

3) $D^b(C)$ is indecomp. if $C$ curve $g \geq 2$

If $X$ = Fano of index $r$, $-K_X = rH$

$\Rightarrow \mathcal{O}_X(-(r-1)H), \dots, \mathcal{O}_X(-H), \mathcal{O}_X$

is exceptional collection

## Fact

- If $K_X$ or $-K_X$ ample $\Rightarrow D^b(X)$ determines $X$ uniquely.
- On the other hand, one can find pairs of abelian vars $X \not\simeq Y$ with $D^b(X) \simeq D^b(Y)$

## More Fano Examples

4) $X = Q_1 \cap Q_2 \subset \mathbb{P}^5$ smooth intersection of 2 quadrics

$X$ is Fano of index $2$

$\Rightarrow \mathcal{O}_X(-H), \mathcal{O}_X$ is exceptional coll.

In fact

$$D^b(X) = \langle \mathcal{O}_X(-H), \mathcal{O}_X, \mathcal{A} \rangle$$

where $\mathcal{A} \simeq D^b(C)$

$$\begin{array}{ll} C & \text{ramified at 6 pts} \\ {\scriptstyle 2:1}\downarrow & \text{where the intersection} \\ \mathbb{P}^1 & \text{of } tQ_1, (t-1)Q_2 \text{ is singular} \end{array}$$

Moreover, $X$ = moduli of rank 2 bundles on $C$.

5) $D^b(X)$ for $X$ cubic 3-fold (smooth) Also Fano index 2

$$D^b(X) = \langle \mathcal{O}_X(-H), \mathcal{O}_X, \mathcal{B} \rangle$$

where $\mathcal{B} \not\simeq D^b(Y)$ for any variety $Y$ !

<u>Notation</u>: One calls $\mathcal{A}$, $\mathcal{B}$ the corresponding "Kuznetsov component" of $X$
$:= Ku(X)$

It should tell us something about rationality of $X$
( Note 4) is rational, 5) is irrational )

<u>Idea</u>:
One can try to define birational invariants associated to the S.O.D.'s

The idea is that for

$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y \end{tikzcd}$$

$f, g$ birational morphisms with $f^*K_X \sim g^*K_X$

<u>Conjecture</u>: In this case $D^b(X) \simeq D^b(Y)$

(If $f^*K_X - g^*K_Y$ is effective, one should have $D^b(Y) \hookrightarrow D^b(X)$)

Known for:
- Standard flip / flop
- GIT flip / flop

⚠ Problem with defining such invariants:

S.O.D.'s are <u>not</u> ~~well defined~~ unique

e.g. can have S.O.D.'s of different length.

<u>Even worse</u>: Can have <u>phantoms</u>

A phantom is $\mathcal{A} \subset D^b(X)$ (admissible) that <u>disappears</u> on the Grothendieck group:

$\mathcal{A} \neq 0$ yet $K_0(\mathcal{A}) = 0$

<u>e.g.</u> (Krah) $X = Bl_{10}\mathbb{P}^2$ blow-up on 10 gral pts

$$\Rightarrow D^b(X) = \underbrace{\langle F_1, \dots, F_{13} \rangle}_{\text{full exc. coll}}$$

$$= \langle E_1, \dots, E_{13}, \mathcal{A} \rangle \quad E_1, \dots, E_{13} \text{ is exc. coll, } \underline{\text{not}} \text{ full}$$

where $\mathcal{A}$ is a phantom

- $D^b(X)$ is an invariant strictly finer than $K_0(X)$.

Most relationships between rationality and derived categories remain conjectural in general.

Examples

- $X$ = cubic 4-fold

$$D^b(X) = \langle \mathcal{O}(-2H), \mathcal{O}(-H), \mathcal{O}, Ku(X) \rangle$$

Conjecture (Kuznetsov): $X$ rational $\iff Ku(X) \simeq D^b(S)$

$S$ = some K3 surface (smooth)

- Conjecture (Orlov): $X$ has full exceptional collection $\implies X$ is rational

- Conjecture: $X$ rational homogeneous space $\implies$ has full exc. coll.