

Derived Categories & S.O.D's

$X = \text{smooth, proj, } k = \mathbb{C}$

Def. $D^b(X) = \left\{ 0 \rightarrow F_i \rightarrow F_{i+1} \rightarrow \dots \rightarrow 0 \right\}$ eventually zero
 $\text{Coh } X$

+ quasi-isomorphisms are invertible

e.g. $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ exact in $\text{Coh } X$

$\Rightarrow A \simeq [B \rightarrow C]^{(0)}$ in $D^b(X)$

$C \simeq [A \rightarrow B]^{(-1)}$

Rmk

$\text{Coh } X = \text{Abelian Category}$ (has short exact seq's)

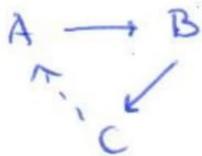
$D^b(X) = \text{Triangulated Category}$ (has exact triangles)

If $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ exact in $\text{Coh}(X)$

$\Rightarrow A \rightarrow B \rightarrow C \rightarrow A[1]$ exact triangle in $D^b(X)$

(shift the complex to the left)

Sometimes write as:



Remark

(1) $D^b(X)$ is the natural environment for sheaf cohomology & other derived functors

e.g. $\Gamma: \text{Coh } X \longrightarrow \text{Vect}_k$ functor of global sections

It is left exact

\Rightarrow Define its right derived functor, which is

$$R\Gamma: D^b(X) \longrightarrow D^b(\text{Vect}_k)$$

which is to apply Γ to a resolution,

say
$$\mathcal{F} \simeq [I^0 \rightarrow I^1 \rightarrow \dots]$$

where $\mathcal{F} \in \text{Coh } X$ and I^i are injectives. The map is a quasi-isomorphism.

$$\& H^i := R^i \Gamma: D^b(X) \longrightarrow \text{Vect}_k$$

is the i -th derived functor

i.e. take i -th cohomology of the resulting chain complex in $D^b(\text{Vect}_k)$

Similarly:

$$\text{e.g. } - \otimes \mathcal{F}: \text{Coh } X \longrightarrow \text{Coh } X$$

It is right exact

\Rightarrow Define its left derived functor

by taking left resolutions by vector bundles

⚠ (injectives are not coherent, and locally free bdl's are not projective, but one can show that these derived functors can be defined and work fine...)

(2) $D^b(X)$ is studied in Mirror Symmetry
ie $D^b(X)$ should be = Fukaya Cat. of some Y
where Y is a symplectic manifold

(3) Bridgeland Stability conditions on $D^b(X)$
allow us to study certain interesting moduli spaces
(e.g. say bundles over a K3) whose entire
birational geometry is encoded in variation of
the stability conditions.

(4) $D^b(X)$ is supposed to tell us something
about birational geometry of X .
There are conjectures of rationality
& possible birational invariants.

We will say more about (4).

First: to study $D^b(X)$ we want to "break
it into pieces", so:

Def:

$D^b(X) = \langle \mathcal{A}_1, \dots, \mathcal{A}_m \rangle$ is a Semi-Orthogonal Decomposition = S.O.D.

if

i) Each $\mathcal{A}_j \subset D^b(X)$ is an "admissible" subcategory

i.e. the inclusion $\mathcal{A}_j \hookrightarrow D^b(X)$ has left & right adjoints \leftarrow

ii) There is no morphisms from right to left:

$$\text{Hom}_{D^b(X)}(\mathcal{A}_j, \mathcal{A}_k) = 0 \quad j > k$$

(basically means $\mathbb{R}\text{Hom}$, i.e. Ext)

iii) They generate $D^b(X)$:

i.e. $D^b(X)$ is the smallest triangulated subcategory containing all $\mathcal{A}_1, \dots, \mathcal{A}_m$

Obs:

(iii) means every $F \in D^b(X)$ is an iterated extension:

$$0 = F_m \longrightarrow F_{m-1} \longrightarrow \dots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F$$

with $A_j \in \mathcal{A}_j$

eg. say $D^b(X) = \langle \mathcal{A}, B \rangle$ is S.O.D.

(i) says $\mathcal{A} \xleftarrow{i} D^b(X)$

has (left) adjoint a

Let $F \in D^b(X)$

By (iii) $\exists A \in \mathcal{A}, B \in B$

$$\text{s.t. } B \rightarrow F \rightarrow A \rightarrow B[1] \quad (*)$$

Here can put $A = a(F)$

Similar for B .

In general, if $\mathcal{A} \subset D^b(X)$ is admissible,

can write $D^b(X) = \langle \mathcal{A}, {}^\perp \mathcal{A} \rangle$

where ${}^\perp \mathcal{A} := \{ \text{Hom}(-, \mathcal{A}) = 0 \}$.

Observe in (*), $B \rightarrow F \rightarrow A$ is indeed zero from (ii).

Remark

• E is exceptional $\Leftrightarrow \langle E \rangle$ is admissible
i.e. $R\text{Hom}(E) = \mathbb{C}$

• E_1, \dots, E_m is exceptional collection

$\Leftrightarrow D^b(X) = \langle E_1, \dots, E_m, B \rangle$ is S.O.D.

where $B :=$ (left) orthogonal to $\langle E_1, \dots, E_m \rangle$

• E_1, \dots, E_m is full $\Leftrightarrow B = 0$.

Examples

- 1) $D^b(\mathbb{P}^n) = \langle \mathcal{O}(-n), \dots, \mathcal{O} \rangle$
- 2) $D^b(X)$ is indecomposable if $K_X = 0$.
- 3) $D^b(C)$ is indecomp. if C curve $g \geq 2$

If $X = \text{Fano}$ of index r , $-K_X = rH$

$$\Rightarrow \mathcal{O}_X(-(r-1)H), \dots, \mathcal{O}_X(-H), \mathcal{O}_X$$

is exceptional collection

Fact

- If K_X or $-K_X$ ample $\Rightarrow D^b(X)$ determines X uniquely.
- On the other hand, one can find pairs of abelian vars $X \not\cong Y$ with $D^b(X) \simeq D^b(Y)$

More Fano Examples

- 4) $X = Q_1 \cap Q_2 \subset \mathbb{P}^5$ smooth intersection of 2 quadrics

X is Fano of index 2

$\Rightarrow \mathcal{O}_X(-H), \mathcal{O}_X$ is exceptional coll.

In fact

$$D^b(X) = \langle \mathcal{O}_X(-H), \mathcal{O}_X, \mathcal{A} \rangle$$

where $\mathcal{A} \simeq D^b(\mathbb{C})$

\mathbb{C} ramified at 6 pts
 $z: \mathbb{C} \rightarrow \mathbb{P}^1$ where the intersection
of $tQ_1, (t-1)Q_2$ is singular

Moreover, $X =$ moduli of rank 2 bundles on \mathbb{C} .

5) $D^b(X)$ for X cubic 3-fold (smooth) Also Fano index 2

$$D^b(X) = \langle \mathcal{O}_X(-H), \mathcal{O}_X, \mathcal{B} \rangle$$

where $\mathcal{B} \not\cong D^b(Y)$ for any variety Y !

Notation: One calls \mathcal{A}, \mathcal{B} the corresponding
"Kuznetsov component" of X
 $:= Ku(X)$

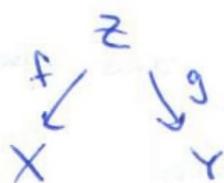
It should tell us something about
rationality of X

(Note 4) is rational, 5) is irrational)

Idea:

One can try to define birational invariants
associated to the S.O.D.'s

The idea is that for



f, g birational morphisms

with $f^*K_X \sim g^*K_Y$

Conjecture: In this case $D^b(X) \simeq D^b(Y)$

(If $f^*K_X - g^*K_Y$ is effective, one should have $D^b(Y) \hookrightarrow D^b(X)$)

Known for:

- Standard flip/flop
- GIT flip/flop

⚠ Problem with defining such invariants:

S.O.D.'s are not ~~well defined~~ unique

e.g. can have S.O.D.'s of different length.

Even worse: Can have phantoms

A phantom is $\mathcal{A} \subset D^b(X)$ (admissible)

that disappears on the Grothendieck group:

$$\mathcal{A} \neq 0 \text{ yet } K_0(\mathcal{A}) = 0$$

e.g. $X = \text{Bl}_{10} \mathbb{P}^2$ blow-up on 10 pt
(Krah)

$$\Rightarrow D^b(X) = \langle F_1, \dots, F_{13} \rangle \quad \text{full exc. coll}$$

$$= \langle E_1, \dots, E_{13}, \mathcal{A} \rangle \quad \begin{array}{l} E_1, \dots, E_{13} \text{ is} \\ \text{exc. coll,} \\ \text{not full} \end{array}$$

where \mathcal{A} is a phantom

- $D^b(X)$ is an invariant strictly finer than $K_0(X)$.

Most relationships between rationality and derived categories remain conjectural in general.

Examples

- $X =$ cubic 4-fold

$$D^b(X) = \langle \mathcal{O}(-2H), \mathcal{O}(-H), \mathcal{O}, K_X(X) \rangle$$

Conjecture: X rational $\Leftrightarrow K_X(X) \simeq D^b(S)$

(Kuznetsov)

$\Rightarrow S =$ some K3 surface
(smooth)

- Conjecture: X has full exceptional collection
(Orlov) $\Rightarrow X$ is rational

- Conjecture: X rational homogeneous space
 \Rightarrow has full exc. coll.