# Fourier-Mukai transforms

Recall some functors between derived categories:

(always $X, Y$ smooth proj, $k = \mathbb{C}$)

Say $f : X \longrightarrow Y$

$\Rightarrow f^* : \mathrm{Coh}\,Y \longrightarrow \mathrm{Coh}\,X$

induces $Lf^* : D^b(Y) \longrightarrow D^b(X)$

Similarly $Rf_* : D^b(X) \longrightarrow D^b(Y)$ if $f$ proper

Also take $\pi : X \longrightarrow pt = \mathrm{Spec}\,k$

$$R\pi_* = R\Gamma : D^b(X) \longrightarrow D^b(pt) = D^b(\mathrm{Vect}_{f.d.} k)$$

$$R^i\Gamma = H^i(X, -)$$

Projection formula: $Rf_*(Lf^*A \overset{L}{\otimes} B) = A \overset{L}{\otimes} Rf_* B$

Adjunction: $\mathrm{Hom}_{D^b(X)}(Lf^*A, B) = \mathrm{Hom}_{D^b(Y)}(A, Rf_*B)$

Definition

For $\mathcal{P} \in D^b(X \times Y)$

$$\begin{tikzcd} & X \times Y \arrow[dl, "q"'] \arrow[dr, "p"] & \\ X & & Y \end{tikzcd}$$

$$\Phi_{\mathcal{P}} := Rp_*(Lq^*(-) \overset{L}{\otimes} \mathcal{P}) : D^b(X) \longrightarrow D^b(Y)$$

is the Fourier-Mukai functor with kernel $\mathcal{P}$

* From now on, we omit $R, L$ from notation but we always mean derived functors.

## Examples

i) $\mathrm{id} = \Phi_{\mathcal{O}_\Delta}$ where $\Delta : X \hookrightarrow X \times X$ diagonal

Indeed, $\Phi_{\mathcal{O}_\Delta}(\mathcal{F}^\cdot) = p_*(q^*(\mathcal{F}^\cdot) \otimes \Delta_* \mathcal{O}_X)$

$$= \underbrace{p_* \Delta_*}_{\mathrm{id}}(\underbrace{\Delta^* q^*}_{\mathrm{id}}(\mathcal{F}^\cdot) \otimes \mathcal{O}_X) = \mathcal{F}^\cdot$$

ii) $f_* = \Phi_{\mathcal{O}_{\Gamma_f}}$ where $\Gamma_f : X \longrightarrow X \times Y$

is the graph

iii) $f^* = \Phi_{\mathcal{O}_{\Gamma_f}} : D^b(Y) \longrightarrow D^b(X)$

iv) $T : D^b(X) \longrightarrow D^b(X)$

$\mathcal{F} \mapsto \mathcal{F}[1]$

$T = \Phi_{\mathcal{O}_\Delta[1]}$

v) $D^b(X) \xrightarrow[(-)\otimes L]{\sim} D^b(X)$ L line bundle

$\shortparallel$

$\Phi_{\Delta_* L}$

vi) serre functor $S : D^b(X) \longrightarrow D^b(X)$

$S = (-) \otimes \omega_X [n] \qquad n = \dim X$

$= \Phi_{\Delta_* \omega_X[n]}$

It is an important functor because it induces $\mathrm{Hom}(A,B) \simeq \mathrm{Hom}(B, S(A))^\vee$

functorially

Lemma

Let $\mathcal{P} \in D^b(X \times Y)$, $Q \in D^b(Y \times Z)$

$\Rightarrow \Phi_Q \circ \Phi_{\mathcal{P}} = \Phi_R$ with $R = (\pi_{XZ})_* (\pi_{XY}^* \mathcal{P} \otimes \pi_{YZ}^* Q)$

- Obs 1 :

Let $E \in D^b(X)$, consider

$\Phi_E = (\ ) \otimes E : D^b(pt) \longrightarrow D^b(X)$

with $E \in D^b(pt \times X) = D^b(X)$

It is fully faithful iff

$$k = R\mathrm{Hom}(k,k) \stackrel{!}{=} R\mathrm{Hom}(\Phi_E(k), \Phi_E(k)) = R\mathrm{Hom}(E,E)$$

which is precisely when $E$ is <u>exceptional</u>

So $E \in D^b(X)$ exceptional

$$\Longleftrightarrow \quad D^b(pt) \simeq \langle E \rangle \hookrightarrow D^b(X)$$

- <u>Obs 2</u>:

Let $\pi : X \to Y$ proper with $R\pi_* \mathcal{O}_X = \mathcal{O}_Y$

then $\pi^* : D^b(Y) \longrightarrow D^b(X)$ is fully faithful

Indeed:

$$R\mathrm{Hom}(\pi^* A, \pi^* B) = R\mathrm{Hom}(A, \pi_*\pi^* B) \quad \text{(adjunction)}$$

$$\begin{aligned} \text{But} \quad \pi_*\pi^* B &= \pi_*(\pi^* B \otimes \mathcal{O}_X) \quad \text{(projection)} \\ &= B \otimes \pi_* \mathcal{O}_X \\ &= B \end{aligned}$$

so $R\mathrm{Hom}(\pi^* A, \pi^* B) = R\mathrm{Hom}(A, B)$

<u>Theorem</u> (Orlov)

If $F : D^b(X) \longrightarrow D^b(Y)$ is fully faithful

$\Longrightarrow F = \Phi_{\mathcal{P}}$ is a Fourier-Mukai functor for some $\mathcal{P} \in D^b(X \times Y)$

Theorem (Bondal-Orlov)

Let $\mathcal{P} \in D^b(X \times Y)$ and $\Phi_{\mathcal{P}} : D^b(X) \to D^b(Y)$

Then $\Phi_{\mathcal{P}}$ is fully faithful $\Longleftrightarrow$

$\forall\ x, y \in X$ closed pts

$$\mathrm{Hom}_{D^b(Y)}\left( \Phi_{\mathcal{P}}(\mathcal{O}_x), \Phi_{\mathcal{P}}(\mathcal{O}_y)[j]\right) = \begin{cases} k & x=y,\ j=0 \\ 0 & x=y,\ j \notin [0, \dim X] \\ 0 & x \neq y \end{cases}$$

($\mathcal{O}_x$ = skyscraper at $x$)

Note: Here $\Phi_{\mathcal{P}}(\mathcal{O}_x) = \mathcal{P}|_{\{x\} \times Y}$ is not an exceptional object of $D^b(Y)$.
Its $\mathrm{Ext}^j$ are $0$ for $j \notin [0, \dim X]$
but not in between.
For instance, $\mathrm{Ext}^1_X(\Phi_{\mathcal{P}}(\mathcal{O}_x), \Phi_{\mathcal{P}}(\mathcal{O}_x))$

$$= \mathrm{Ext}^1_X(\mathcal{O}_x, \mathcal{O}_x)$$

$$= T_x X \neq 0$$

How to compute RHom ?

$$\mathrm{Hom}(A, -) = \Gamma \circ \mathcal{H}om(A, -)$$

so $R\mathrm{Hom}(A,B) = R\Gamma \circ R\mathcal{H}om(A,B)$

If $A$ = locally free $\mathcal{S}$

$$\Longrightarrow R\mathcal{H}om(A,B) = \mathcal{H}om(A,B) = A^\vee \otimes B$$

$$\& \quad R\mathrm{Hom}(A,B) = R\Gamma(A^\vee \otimes B)$$

i.e. need to compute $H^i(X, A^\vee \otimes B)$

Moduli spaces

Let $M = M_C(r, L)$ moduli space of vector bundles on $C$ of $rk = r$ & $\det = L$

where $C$ = curve of genus $g \geq 2$

$d = \deg L$, $(d, r) = 1$

$\Rightarrow$ $M$ is Fano of index 2

$\operatorname{Pic} M = \mathbb{Z}$, generated by $\Theta$ = ample

$\Rightarrow$ $D^b(M) = \langle \Theta^\vee, \mathcal{O}_M, \mathcal{A} \rangle$

where $\mathcal{A}$ = Kuznetsov comp.

$M$ carries a universal bundle $\mathcal{U} \in D^b(C \times M)$

$\rightsquigarrow$ define $\Phi_{\mathcal{U}} : D^b(C) \longrightarrow D^b(M)$

Thm (Belmans-Mukhopadhyay, Lee-Moon)

$\Phi_{\mathcal{U}} : D^b(C) \hookrightarrow D^b(M)$ is fully faithful

Moreover, if $g \geq 6$

$$D^b(M) = \langle \Theta^\vee, \Theta^\vee \otimes \Phi_{\mathcal{U}}(D^b(C)), \mathcal{O}_M, \Phi_{\mathcal{U}}(D^b(C)), \mathcal{A}' \rangle$$

$$= \langle D^b(pt), D^b(C), D^b(pt), D^b(C), \mathcal{A}' \rangle$$

is a S.O.D.

When $r = 2$, there is a s.o.d. as follows:

Thm (Tevelev-T., Tevelev)

For $r = 2$, $$D^b(M) = \left\langle \begin{matrix} D^b(\operatorname{Sym}^\alpha C), D^b(\operatorname{Sym}^\alpha C) \\ D^b(\operatorname{Sym}^{g-1} C) \end{matrix} \right\rangle_{\alpha = 0, \dots, g-2}$$

The functors $D^b(\mathrm{Sym}^\alpha C) \longrightarrow D^b(M)$ are obtained as follows:

$$\begin{tikzcd} C \times \dots \times C \arrow[r, "\tau"] \arrow[d, "\pi_j"', "\text{j-th projection}"] & \mathrm{Sym}^\alpha C \\ C \end{tikzcd}$$

Let $\tau$ quotient map by $S_\alpha$

$\Longrightarrow$ $\bigotimes_{j=1}^{\alpha} \pi_j^* \mathcal{U}$ has $S_\alpha$-equivariant structure

$$\mathcal{U}_\alpha := \tau_*^{S_\alpha}\left(\bigotimes_{j=1}^{\alpha} \pi_j^* \mathcal{U}\right) \quad S_\alpha\text{-invariant pushfwd}$$

$$\in D^b(\mathrm{Sym}^\alpha C \times M)$$

gives $\Phi_{\mathcal{U}_\alpha} : D^b(\mathrm{Sym}^\alpha C) \longrightarrow D^b(M)$.

To prove that $\Phi_{\mathcal{U}_\alpha}$ is fully faithful, one needs to study

$$\mathrm{Ext}^j\left(\Phi_{\mathcal{U}_\alpha}(\mathcal{O}_{\{D\}}), \Phi_{\mathcal{U}_\alpha}(\mathcal{O}_{\{D'\}})\right)$$

(for $D, D' \in \mathrm{Sym}^\alpha C$ degree $\alpha$ divisors)

$$= H^j\left(M, \mathcal{U}_\alpha^\vee\big|_{\{D\}\times M} \otimes \mathcal{U}_\alpha\big|_{\{D'\}\times M}\right)$$

<u>Thm</u> (Lee-Moon)

For $r > 2\alpha$, $\Phi_{\mathcal{U}_\alpha}$ is fully faithful

i.e. $D^b(\mathrm{Sym}^\alpha C) \hookrightarrow D^b(M)$.

($g \geq 2$, $(d,r) = 1$ as always)