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Statistical Methods for detecting changes in a Linear Dynamical Systems

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Abstract

The problem of detecting changes in the parameters of a dynamical system through statistical techniques is considered. In particular, a rotor is modelled as a stochastic linear dynamical system. An instrumental test for fault detection in its normal performance is developed. Computational simulations are carried out.

1. Introduction

The problem of fault detection in dynamical systems has received a lot of attention, in many fields of application. In this paper we apply some modern techniques for modelling and detecting small changes in the eigenstructure of a rotor. Commonly, Spectral Analysis has been used in the study and monitoring of rotational machines when structural faults arise. Spectral methods are widely used in this context and can be called "nonparametric methods" because no statistical hypothesis are made in its implementation, [7]. However, in recent years several improvements have been carried out, motivated by the increasing availability of computer power. A number of parametric methods based on time domain have been developed on this subject. Statistical techniques such as maximum likelihood methods and linear stochastic system should be mentioned in this respect. These new techniques are based on the partial knowledge of the internal structure of dynamic phenomena and involve a sophisticated signal modelling. State space equations have been used for describing rotational systems and parametric statistical models –such as Markov–Gaussian process– have been studied for describing input–output signals. In this way, optimal processing of sensor information can be made.

The objectives of this work are basically two: the modelling of a rotor as a stochastic linear dynamical system and the developing of a statistical test for failure detection.

2. Description of mechanic system and modelling

The system that we are considering is a rotor, Fig. 1. The torsional vibration of a rotor can be modelled as a mechanic system which obeys the following equations:

$$\left| \begin{array}{l} \ddot{\alpha}_n(t) + 2\sigma w_n \dot{\alpha}_n(t) + w_n^2 \alpha_n(t) = \sqrt{f_n} N_{1n}(t) \quad n = 1, \dots 5 \\ \alpha_n(0) = \alpha_{no} \quad \dot{\alpha}_n(0) = \dot{\alpha_{no}} \end{array} \right. \tag{1}$$

where α_n is the angular position corresponding to n–oscilation mode and w_n its eigenfrequency, σ is the damping coefficient, and $N(t)$ is the external excitation assumed to be Gaussian white noise with variance f_n.

The observation vector $Y(t)$ contains the accelerometer information, which is described by

$$Y_j(t) = \sum_{k=1}^{5} (\phi_k(b_j) - \phi_k(a_j)) \ddot{\alpha}_k(t) + \sqrt{g_j} N_{2j}(t) \quad j = 1, 2 \tag{2}$$

where $\phi_k(\xi) = (\sqrt{2}/L)\cos(k\pi/L\xi)$; L is the length of rotor; a_j, b_j are the positions of acelerometers, and $N_{2j}(t)$ is Gaussian white noise with variance g_j.

The system described by (1) and (2) is equivalent to the following continuos–time state space model:

$$\left| \begin{array}{l} \dot{X}(t) = AX(t) + FN(t) \\ X(0) = X_0 \\ Y(t) = CX(t) + GN(t) \end{array} \right. \tag{3}$$

where:

$$\begin{array}{rcl} X & = & (\alpha_1, \alpha_2, \cdots, \alpha_5, \dot{\alpha}_1, \dot{\alpha}_2, \cdots, \dot{\alpha}_5)^T \\ N & = & (N_{11} N_{12} \cdots, N_{15}, N_{21}, N_{22})^T \end{array}$$

$$A = \left|\begin{array}{cccccccccc} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -w_1^2 & 0 & 0 & 0 & 0 & -2\sigma w_1 & 0 & 0 & 0 & 0 \\ 0 & -w_2^2 & 0 & 0 & 0 & 0 & -2\sigma w_2 & 0 & 0 & 0 \\ 0 & 0 & -w_3^2 & 0 & 0 & 0 & 0 & -2\sigma w_3 & 0 & 0 \\ 0 & 0 & 0 & -w_4^2 & 0 & 0 & 0 & 0 & -2\sigma w_4 & 0 \\ 0 & 0 & 0 & 0 & -w_5^2 & 0 & 0 & 0 & 0 & -2\sigma w_5 \end{array}\right|$$

$$F = \left|\begin{array}{ccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{f_1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{f_2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{f_3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{f_4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{f_5} & 0 & 0 \end{array}\right|$$

$$Y = (y_1, y_2)^T$$

$$C = \left[\begin{array}{cccccccc} -c_{11}w_1^2 & -c_{12}w_2^2 & \cdots & -c_{15}w_5^2 & -c_{11}2\sigma w_1 & \cdots & -c_{15}2\sigma w_5 \\ -c_{21}w_1^2 & -c_{22}w_2^2 & \cdots & -c_{25}w_5^2 & -c_{21}2\sigma w_5 & \cdots & -c_{25}2\sigma w_5 \end{array}\right]$$

with $c_{ij} = \phi_j(b_i) - \phi_j(a_1)$

$$G = \left[\begin{array}{cccccc} c_{11}\sqrt{f_1} & c_{12}\sqrt{f_2} & \cdots & c_{15}\sqrt{f_5} & \sqrt{g_1} & 0 \\ c_{21}\sqrt{f_1} & c_{22}\sqrt{f_2} & \cdots & c_{25}\sqrt{f_5} & 0 & \sqrt{g_2} \end{array}\right]$$

The c_{ij} values are determined by the experimental design. In this case

$$(c_{ij}) = \left[\begin{array}{ccccc} -2 & 0 & -2 & 0 & -2 \\ -1/2 & -3/2 & -2 & -3/2 & -1/2 \end{array}\right]$$

2.1 Properties of state space representation

Stability, observability and controllability are basic concepts in Control Theory. In loose terms, stability means that the term in the solution to (3), associated to the initial conditions, will asymptotically vanish.

Also, a state is controlable if it can be reached from the zero state in some finite number of steps by an appropiate input. Finally, the dynamic system is observable if X_0 can be determined from observations, [3].

The conditions above are fundamental for system performance and test implementation.

In our case, the dynamic system is stable, controlable and observable.

2.2 Discrete–Time equivalent system

Since a digital computer is intrinsically a discrete-time system and due to sampling needs, we present the discrete time equivalent model:

$$\left.\begin{array}{rcl} X_{n+1} & = & A'X_n + F'N_n \\ Y_n & = & CX_n + GN_n \end{array}\right| \tag{4}$$

where $A' = e^{\Delta A}$, $F' = \sqrt{\Delta}F$ and Δ = sampling interval.

This discrete–time system is stable, controlable and observable if the sampling interval Δ is such that

$\mathrm{Im}[\lambda_i - \lambda_j] \neq 2\pi n/\Delta$, whenever $\mathrm{Re}[\lambda_i - \lambda_j] = 0$, where $[\lambda_i]$ are the eigenvalues of A, [5].

2.3 2–Modes modelling

In many mechanic applications a reduced model is proposed, that considers only the first two oscilation modes. We will also analyze this simplified model.

In this way, the model is reduced to:

$$\left|\begin{array}{l} \dot{x}(t) = Bx(t) + FN(t) \\ x(0) = x_0 \\ y(t) = Dx(t) + GN(t) \end{array}\right. \tag{5}$$

where

$$x = \begin{vmatrix} \alpha_1 \\ \alpha_2 \\ \dot{\alpha}_1 \\ \dot{\alpha}_2 \end{vmatrix} \qquad N = \begin{vmatrix} n_{11} \\ n_{12} \\ n_{21} \\ n_{22} \end{vmatrix}$$

$$A = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -w_1^2 & 0 & -2\sigma w_1 & 0 \\ 0 & -w_2^2 & 0 & -2\sigma w_2 \end{vmatrix} \quad F = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sqrt{f_1} & 0 & 0 & 0 \\ 0 & \sqrt{f_2} & 0 & 0 \end{vmatrix}$$

$$C = \begin{bmatrix} -c_{11}w_1^2 & -c_{12}w_2^2 & -c_{11}2\sigma w_1 & -c_{12}2\sigma w_2 \\ -c_{21}w_1^2 & -c_{22}w_2^2 & -c_{21}2\sigma w_1 & -c_{22}2\sigma w_2 \end{bmatrix}$$

$$G = \begin{bmatrix} c_{11}\sqrt{f_1} & c_{12}\sqrt{f_2} & \sqrt{g_1} & 0 \\ c_{21}\sqrt{f_1} & c_{22}\sqrt{f_2} & 0 & \sqrt{g_2} \end{bmatrix}$$

2.4 Explicit Solution of Continuos–Time State Equations

Continuous–time state equations have an explicit solution, see [4]. In this case, the linear dynamical system obeys the state space equations:

$$\left| \begin{array}{l} X(t) = AX(t) + FN(t) \\ X(0) = X_0 \\ Y(t) = CX(t) + GN(t) \end{array} \right. \tag{6}$$

It is clear that it suffices to find a solution for $X(t)$ since $Y(t)$ is a linear function of the state. The solution at t of the above differential equations with initial condition X_0 at $t = 0$ is:

$$X(t) = \exp(At)X_0 + \int_0^t \exp[A(t-s)]FN(s)ds \tag{7}$$

Therefore, an explicit solution for $Y(t)$ is:

$$Y(t) = C\exp(At) + \int_0^t C\exp[A(t-s)]F\,N(s)ds + G\,N(t) \tag{8}$$

In this context, the $\exp(At)$ expression means

$$\exp(At) = \sum_{n=0}^{\infty}(At)^n/n!$$

2.5 Statistical behavior of state and observation realizations

Assuming that $E[X(0)] = \mu$, the expected value of $X(t)$ is:

$$E[X(t)] = \mu\exp(At)$$

Because the system is stable, $\mu \exp(At) \to 0$ when $t \to \infty$, hence $E[X(t)] \to 0$ when $t \to \infty$. In fig. 2 we can see the approximate behavior of the state through time.

Let $\overline{X}(t) = X(t) - E[X(t)]$, then the state variance matrix is given by the following expression:

$$E[\overline{X}(t)\overline{X}(t)^T] = \exp(At)\Delta_0 \exp(A^T t) + \int_0^t \exp(As)FF^T \exp(A^T s)ds \tag{9}$$

Let $R_0(t) = \exp(At)\Delta_0 \exp(At)$ the initial variance, i.e. the variance due to the initial state, and let

$$R_1(t) = \int_0^t \exp(As)FF^T \exp(A^T s)ds \tag{10}$$

the variance due to the noise. We can then write the following variance decomposition:

$$\text{Var}\,[X(t)] = R_0(t) + R_1(t) \tag{11}$$

When the system is (A, F) controlable, the matrix $R_1(t)$ is not singular. This means that the noise is propagated to all components of the state vector. Furthermore, if the system is stable, then $R_0(t)$ tends to zero when t tends to infinity and then $R(t)$ tends to the covariance matrix R_∞ defined by:

$$R_\infty = \int_0^\infty \exp(As)FF^T \exp(A^T s)ds \tag{12}$$

i.e. the state variance matrix decreases asympotically to R_∞.

Fig. 3 and Fig. 4 show the asymptotic behavior of R_0 and R_1 respectively.

3. Global test for detecting changes

An important and recently developed test for detecting changes in dynamical systems is described in [4]. We have adapted this procedure to the problem of fault detection in the rotor. The basic idea involved in the test is to transform the state space representation into a multidimensional process ARMA and then use the generalized likelihood ratio test χ^2 for detecting changes in the mean of a Gaussian process with known covariance matrix. An advantage of this test is its robustness with respect to possible nonstationary excitations.

3.1. Global test

Let the dynamic system be described by

$$\left| \begin{array}{l} X_{t+1} = FX_t + W_t \\ y_t = HX_t \end{array} \right. \tag{13}$$

with $\dim(X) = n$, $\dim(Y) = r$ and W_t white noise with $\text{Cov}\ (W_t) = Q_t$.

Consider the multidimensional process ARMA$(p, p-1)$ defined by

$$Y_t = \sum_{k=1}^{p} A_k Y_{t-k} + \sum_{j=1}^{p-1} B_j(t) E_{t-j} \tag{14}$$

where A_k is an autorregressive $r \times r$ matrix, B_j is a moving average $r \times r$ matrix, and E_t is a Gaussian white noise with identity covariance matrix.

It is shown in [1], that models (13) and (14) are equivalent. Furthermore, model (14) can be obtained directly from model (14) by solving the following linear system of equations:

$$HF^p = \sum_{i=1}^{p} A_i H F^{p-i} \tag{15}$$

$$B_j = H(F^j + A_j F^{j-1} + \cdots + A_j F + A_j) \tag{16}$$

3.2. Instrumental statistic

Let

$$U_N(s) = \sum_{t=1}^{s} Z_t W_t^T \tag{17}$$

where $Z_t^T = (Y_{t-p}^T, \cdots, Y_{t-p-N+1}^T)$ is a vector of $N \geq p$ instrumental variables, and where W_t is

$$W_t = Y_t - \sum_{t=1}^{s} A_i Y_{t-1} = Y_t - \theta^T \phi_t \tag{18}$$

with

$$\phi_t^T = (Y_{t-p}^T, \cdots, Y_{t-1}^T) \tag{19}$$

and

$$\theta^T = (A_p, \cdots, A_1) \tag{20}$$

Matrix $U_N(s)$ can be writen in the following form:

$$U_N(s) = \mathcal{H}_{p+1}^T(s) \begin{pmatrix} \theta^0 \\ -I_r \end{pmatrix}, \tag{20}$$

where $\mathcal{H}_{p+1,N(s)}$ is the empirical Hankel matrix of the observed process (Y_t).

$$\mathcal{H} = \begin{pmatrix} R_0(s) & \cdots & R_{q-1}(s) \\ \vdots & & \vdots \\ R_{p-1}(s) & \cdots & R_{p+q-2}(s) \end{pmatrix} \quad (22)$$

$$R_m(s) = \sum_{t=1}^{s-m} Y_{t+m} Y_t^T \quad (23)$$

Under the hypotheses H_0 of no change, i.e. $\theta = \theta^0$, W_t is a MA process, which is uncorrelated with Z_t, and thus $U_N(s)$ has zero-mean.

On the other hand, under the hypotheses H_1 of small change, i.e. $\theta = \theta^0 + \delta\theta/\sqrt{s}$, the mean of $U_N(s)$ is equal to the mean of:

$$\frac{1}{\sqrt{s}} \mathcal{H}_{p \cdot n}^T(s) \delta\theta \quad (24)$$

Using Kronecker products (see [4]), we can define

$$\text{Var } \theta_n(s) \equiv \text{ COL } (U_N^T(s)) = \sum_{t=1}^{s} Z_t \otimes W_t \quad (25)$$

Under H_0, $U_N(s)$ has covariance matrix $\Sigma_N(s)$ given by

$$\Sigma_N(s) = \sum_{t=1}^{s} \sum_{i=-p+1}^{p-1} E(Z_t Z_{t-i}^T \otimes W_t W_{t-i}^T) \quad (26)$$

For instance, an estimator of Σ_N is:

$$\hat{\Sigma}_N(s) = \sum_{t=1}^{s} \sum_{i=-p+1}^{p-1} (Z_t Z_{t-i}^T \otimes W_t W_{t-i}^T) \quad (27)$$

It is shown in [5], that $(1/s)\hat{\Sigma}_N$ is a consistent estimator of $(1/s)\Sigma_N(s)$ under both hypotheses H_0 and H_1, and that $(1/\sqrt{s})U_N(s)$ is asymptotically Gaussian distributed under both hypotheses.

Because of the mean value of $U_N(s)$ under H_1, none of the changes $\delta\theta$ belonging to the kernel of $(1/s)\mathcal{H}_{p,N}^T(s)$ can be detected.

Now, recall that if a vector U is Gaussian with mean μ and covariance Σ, for testing $\mu = 0$ against $\mu \in$ Range (M) where M is a full column rank matrix, the generalized likelihood ratio test is:

$$U^T \Sigma^{-1} M (M^T \Sigma^{-1} M)^{-1} M^T \Sigma^{-1} U \quad (28)$$

In order to apply this test, it is necessary to reduce

$$M = \mathcal{H}_{p,N}^T(s) \otimes I_r \quad (29)$$

to a full column rank matrix. This is possible because the system is observable. Furthermore, in the case that the number of sensors r divides the state dimension n, the statistic (28) reduces to

$$t_0 = \text{ Var } \theta_p^T(s)\Sigma_p^{-1}(s) \text{ Var } \theta_p(s) \tag{30}$$

3.3. Application to fault detection in a rotor

The state space model for torsional vibration of rotor must be slightly modified to use the global test. Model (3) can be written as:

$$Z_{n+1} = \begin{bmatrix} A' & 0 \\ 0 & \lambda I \end{bmatrix} Z_n + \begin{bmatrix} F & 0 \\ -\lambda G & G \end{bmatrix} W_n \tag{31}$$

$$Y_n = [C \quad I]\, Z_n \tag{32}$$

Model (31) is another state space representation of rotor, where $Z_n = \begin{bmatrix} X_n \\ GN_n \end{bmatrix}$ is the new state vector, $W_n = \begin{bmatrix} N_n \\ N_{n+1} \end{bmatrix}$ is not white noise and λ is a scalar value belonging to interval $(0, 1)$. This parameter must be introduced to preserve stability of the transition matrix.

In a first stage, the process of identification of modal frequencies or damping coefficients can be made by means of Spectral Methods.

In a second stage, in which we are concerned in the detection of changes it is possible to set empirical thresholds for ϵ, the deviation of t_0 statistic from its normal values.

In our application, we assume that eigenfrequencies and damping coefficients are known; therefore we are concerned only with small changes of these modal frequencies.

We applied the global test to model (31) and carried out computational simulations obtaining useful insights on their performance.

3.4. Simulation results

Tables 1, 2 and 3 show the results of the simulations. Table 1 shows results from a 2–modes system, for different changes in its eigenfrequencies over 4500 records. Table 2 shows results from a 5–modes system for different changes in its eigenfrequencies over 800 records of observations. Table 3 shows results from a 2–modes system for 1% of variation in the first eigenfrequency over 4000 records.

The paremeters used in these simulations are:

$$
\begin{array}{ll}
f_n = 10^{-3} & n = 1, \cdots, 5 \\
g_n = 10^{-3} & n = 1, 2 \\
\text{Initial State} & X = 0 \\
\lambda = 0.5 &
\end{array}
$$

In all cases, the instrumental test increases when changes arise. For instance, in Table 3, the behavior of t_0 is extremely regular in both normal and not normal cases.

Results from 5–modes modelling can not be compared with results from 2–modes because they use distinct lenghts of records or differents changes in the eigenfrequencies.

The computational algorithm is stable. In the practice, the instrumental test increases when more data is added. Hence, it is possible to consider windows of $4000 - 5000$ observations.

3.4. Conclusions

In this paper we have presented a state space representation of a specific mechanic system. We have stated that it is a stable–observable–controlable system, its behavior being thereby asymptotically stationary. Furthermore we have applied a global test for detecting changes in the eigenfrequencies of torsional vibration of the rotor. Decision rules for fault detection can be implemented in all cases.

Table 1: Global Test

2–Modes / 4500 observations

Experiment	σ	w_1	w_2	t_0	eps
1	0,001	1,5707	3,1415	740,9	0
2	0,001	1,4921	3,1415	747,6	6,7
3	0,001	1,5707	2,9844	745,4	4,5

Table 2: Global Test

5–Modes / 800 observations

Experiment	σ	w_1	w_2	t_0	eps
1	0,001	1,570	3,1415	46,17	0
2	0,001	5,000	3,1415	61,88	15,71
3	0,001	1,570	6,000	55,46	9,29

Table 3: Global Test

2–Modes / 4000 observations 1% variation in the first frequency

	Ex.N°/Seed	σ	w_1	w_2	t_0
1	123457	10^{-6}	1,5707	3,1415	799,819481
2	123457	10^{-6}	1,5864	3,1415	800,531425
3	7389367	10^{-7}	1,5707	3,1415	799,921031
4	7389367	10^{-7}	1,5864	3,1415	800,432144
5	24081963	10^{-7}	1,5707	3,1415	799,921006
6	24081963	10^{-7}	1,5864	3,1415	800,432304
7	99999999	10^{-7}	1,5707	3,1415	799,920922
8	99999999	10^{-7}	1,5707	3,1415	800,432315
9	20011926	10^{-7}	1,5707	3,1415	799,920925
10	20011926	10^{-7}	1,5864	3,1415	800,432517
11	30000000	10^{-7}	1,5707	3,1415	799,921002
12	30000000	10^{-7}	1,5864	3,1415	800,432132

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Resumen

Se considera el problema de detección de cambios en los parámetros de un sistema dinámico a través de técnicas estadísticas, En particular, se modela un rotor como un sistema dinámico lineal estocástico y se desarrolla un test instrumental para detectar fallas en su funcionamiento normal. Se realizan simulaciones computacionales.