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Computational Statistics & Data Analysis 50 (2006) 568 – 582
COMPUTATIONAL STATISTICS & DATA ANALYSIS
www.elsevier.com/locate/csda

Estimation of seasonal fractionally integrated processes

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Received 4 August 2004; accepted 6 August 2004
Available online 2 September 2004

Abstract

This paper discusses the estimation of fractionally integrated processes with seasonal components. In order to estimate the fractional parameters, we propose several estimators obtained from the regression of the log-periodogram on different bandwidths selected around and/or between the seasonal frequencies. For comparison purposes, the semi-parametric method introduced in Geweke and Porter-Hudak (1983) and Porter-Hudak (1990) and the maximum-likelihood estimates (ML) are also considered. As indicated by the Monte Carlo simulations, the performance of the estimators proposed is good even for small sample sizes.

Keywords: Fractional differencing; Long memory; Maximum-likelihood estimates; Periodogram regression; Seasonality

1. Introduction

The fractionally integrated autoregressive moving average model (ARFIMA) with seasonal component has recently been considered in many works and has been used to describe a large number of real world cyclical phenomena exhibiting long-range dependence. For example, Porter-Hudak (1990) has examined monetary aggregates using a seasonal

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doi:10.1016/j.csda.2004.08.004

differenced process. This model has been also considered by Hassler (1994). Both works have employed the semi-parametric regression of the log-periodogram method, cf. Geweke and Porter-Hudak (1983), for the estimation of the fractional long-memory parameter in the seasonal ARFIMA model. The spectral density of a non-seasonal ARFIMA process has only one pole at zero frequency. However, some works have recently proposed an extension of the ARFIMA process to model time series with long-memory behavior at any given frequency in $[0,\pi]$. This generalization is the Gegenbauer autoregressive moving average (GARMA) model. The GARMA model was first suggested by Hosking (1981) and later studied by Andel (1986), Gray et al. (1989, 1994) and Chung (1996). Other extensions of the GARMA process are the fractional ARUMA model discussed by Giraitis and Leipus (1995) and the k-factor GARMA models proposed by Woodward et al. (1998). The ARUMA or k-factor GARMA models allow k long-memory parameters associated to k frequencies in $[0,\pi]$. In addition, Arteche and Robinson (2000) have introduced the seasonal or cyclical asymmetric long-memory process and the usual estimation methods, regression of the log-periodogram and local Whittle are extended to this model.

The main objective of this paper is to propose a number of semi-parametric estimation methodologies for seasonal long-range-dependent processes and to evaluate their performance via Monte Carlo simulations. Like the Geweke and Porter-Hudak method, the proposed techniques are based on the regression of the log-periodogram. However, we consider a number of different bandwidth selections around the seasonal frequencies as explained in Section 3. Our Monte Carlo experiments indicate that some of the bandwidth choices produce very good estimates of the long-memory parameters. For comparison purpose, the parametric Gaussian ML estimator, see for example Section 5.3 of Beran (1994), is included in the simulation study.

The remaining of this paper is structured as follows. The Section 2 presents the class of seasonal fractionally integrated processes under consideration in this simulation study. Section 3 addresses several semi-parametric estimation procedures as well as the maximum-likelihood method. Section 4 studies the behavior of the estimation procedures through Monte Carlo simulation and final remarks are given in Section 5.

2. The model

Let $\{X_t\}$ be a zero-mean seasonal fractionally integrated process defined as

$$(1-B)^d(1-B^s)^D X_t = \varepsilon_t \tag{1}$$

for $t=1,\ldots,n$, where $d, D \in \mathbb{R}$, $\{\varepsilon_t\}_{t\in\mathbf{Z}}$ is a white-noise process with zero mean, variance σ_ε^2, B is the usual backshift operator and s is the seasonal period. The process specified by (1) is the seasonal fractionally integrated processes denoted here by ARFISMA$(0,d,0)\times(0,D,0)_s$. A more general class of seasonal models can be generated by allowing $\{\varepsilon_t\}$ to be an stationary and invertible seasonal ARMA process. This generalized process is denoted ARFISMA$(p,d,q)\times(P,D,Q)_s$ where p, q, P and Q are the polynomial order of the non-seasonal and seasonal autoregressive (AR) and moving-average (MA) operators, respectively. However, for convenience, this paper is restricted to time-series models with $p=q=P=Q=0$. Definition (1) is motivated by the works by Peiris and Singh (1996) and

Hassler (1994). The former discussed the general ARFISMA process focusing on prediction issues and the latter considered model (1) with $d=0$ and defined $(1-B^s)^D$ as a *rigid* filter. Hassler (1994) also presented the flexible ARFISMA model. The flexible model was also considered in Ooms (1995). In this work, the author presented a survey related to seasonal long-memory process, discussed the estimation and inference of the parameters and considered some applications.

The spectral density of a non-seasonal ARFIMA model behaves like $f(\lambda) \sim C|\lambda|^{-2d}$, for $\lambda \to 0$, for some positive constant C, and the autocorrelation between X_t and X_{t+k} satisfies $\rho(k) \sim k^{2d-1}$, as $k \to \infty$. When $|d| < \frac{1}{2}$ the process is stationary and invertible. For positive d the process is said to have long memory and, when $d=0$ or $d<0$ the process is said to be short or intermediate memory (antipersistent), respectively. For a detailed review of these processes see, for example, Beran (1994). Since the seminal work by Geweke and Porter-Hudak (1983), many estimators of the parameter d have been proposed in the literature of long-memory time series, see, for example Reisen (1994), Arteche and Robinson (2000) and references therein. A recent empirical investigation of different methods for estimating ARFIMA(p,d,q) models is given in Reisen et al. (2001). Additional references for parameter estimation in long-memory processes include McCoy and Walden (1996), Jensen (1999), Whitcher (2004) and Lopes et al. (2004).

Giraitis and Leipus (1995) have introduced the fractional ARUMA$(0, d_1, \ldots, d_r, 0)$ noise model with parameters $d_1, \ldots, d_r$ $(d_j \neq 0, j=1, \ldots, r)$ and fixed frequencies $0 \leqslant \lambda_1 <, \ldots, < \lambda_r \leqslant \pi$ as a stationary process Y_t obtained from the solution of the equation

$$\nabla_{\lambda_1,\ldots,\lambda_r}^{d_1,\ldots,d_r} Y_t = \varepsilon_t, \tag{2}$$

where ε_t is defined as in Eq. (1), d_j are fractional differencing degrees and $\nabla_{\lambda_1,\ldots,\lambda_r}^{d_1,\ldots,d_r} = \prod_{j=1}^{r}[(1-B\mathrm{e}^{\mathrm{i}\lambda_j})(1-B\mathrm{e}^{-\mathrm{i}\lambda_j})]^{d_j}$. As shown below, model (1) is a particular case of the fractional ARUMA process defined by (2). For simplicity, assume that the period s is even and let λ_v be the seasonal frequency defined as $\lambda_v = \frac{2\pi v}{s}$, for $v = 1, 2, \ldots, \frac{s}{2}$. The filter in Eq. (1) may be written as

$$\begin{aligned}(1-B)^d(1-B^s)^D &= (1-B)^d(1-B)^D(1+B)^D \prod_{j=1}^{\frac{s}{2}-1}[(1-B\mathrm{e}^{\mathrm{i}\lambda_j})(1-B\mathrm{e}^{-\mathrm{i}\lambda_j})]^D \\ &= [(1-B\mathrm{e}^{\mathrm{i}0})(1-B\mathrm{e}^{-\mathrm{i}0})]^{(d+D)/2}[(1-B\mathrm{e}^{\mathrm{i}\pi})(1-B\mathrm{e}^{-\mathrm{i}\pi})]^{D/2} \\ &\quad \times \prod_{j=1}^{\frac{s}{2}-1}[(1-B\mathrm{e}^{\mathrm{i}\lambda_j})(1-B\mathrm{e}^{-\mathrm{i}\lambda_j})]^D \\ &= \prod_{j=0}^{\frac{s}{2}}[(1-B\mathrm{e}^{\mathrm{i}\lambda_j})(1-B\mathrm{e}^{-\mathrm{i}\lambda_j})]^{d_j}\end{aligned} \tag{3}$$

with $d_0 = \frac{d+D}{2}$, $\lambda = 0$, $d_j = D$ for $j = 1, \ldots, \frac{s}{2}-1$ and $d_{\frac{s}{2}} = \frac{D}{2}$.

Using expression (3) and Theorem 2 of Giraitis and Leipus (1995), we conclude that model (1) is an ARUMA$(0, \frac{d+D}{2}, D, \ldots, D, \frac{D}{2}, 0)$ process and it is causal and invertible if

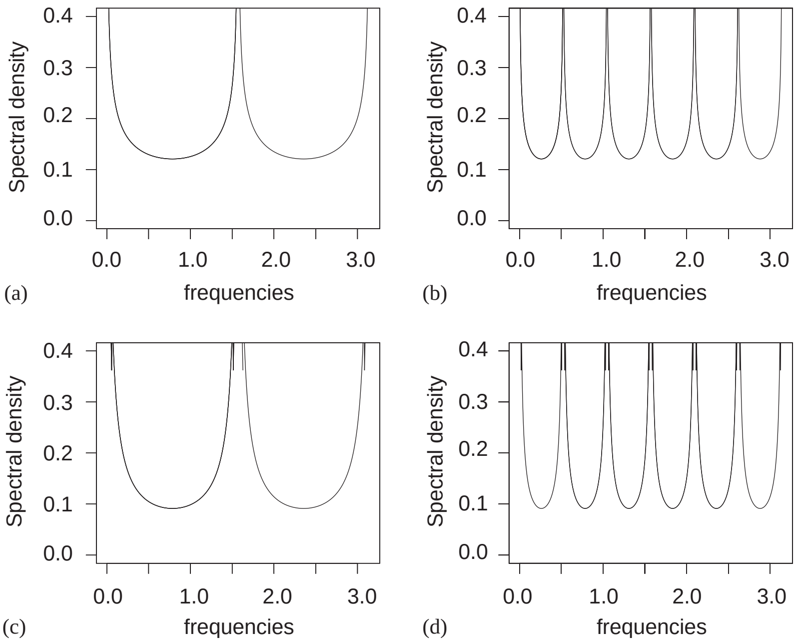

Fig. 1. Spectral density of the model (1) when $d = 0$: (a) model ARFISMA$(0, 0.2, 0)_4$; (b) model ARFISMA$(0, 0.2, 0)_{12}$; (c) model ARFISMA$(0, 0.2, 0)_4$ and (d) model ARFISMA$(0, 0.4, 0)_{12}$.

and only if $|d+D| < \frac{1}{2}$, $|D| < \frac{1}{2}$. Thus, according to expression (3), the seasonal fractionally integrated process defined by (1) has a long-memory component for $d + D > 0$, i.e. it has a pole at zero frequency. Furthermore, model (1) has long-memory seasonal components at frequencies $\lambda_v = \frac{2\pi v}{s}$, for $v = 1, 2, \ldots, \frac{s}{2}$.

Note that for s odd, the expression (3) does not have the term $(1 + B)$. In this situation, the model (1) is an ARUMA$(0, \frac{d+D}{2}, D, \ldots, D, 0)$ model.

By allowing $d, D \in (-\frac{1}{2}, \frac{1}{2})$, the spectral density of model (1) displays zeros or poles at some frequencies in the interval $[-\pi, \pi]$. The spectral density of model (1) is given by

$$f(\lambda) = \frac{\sigma_\varepsilon^2}{2\pi}[2\,\sin(\lambda s/2)]^{-2D}[2\,\sin(\lambda/2)]^{-2d} \tag{4}$$

for $-\pi \leqslant \lambda \leqslant \pi$. Observe that at the seasonal frequencies $\lambda_v = 2\pi v/s$, $v = 0, 1, \ldots, [s/2]$, and for $D > 0$ and/or $d > 0$, $f(\lambda)$ becomes unbounded and behaves as

$$f\left(\lambda + \frac{2\pi v}{s}\right) \sim \frac{\sigma_\varepsilon^2}{2\pi}|\lambda s|^{-2D}\left|2\,\sin\frac{\pi v}{s}\right|^{-2d} \quad \lambda \to 0, \tag{5}$$

where $v > 0$. Observe that the expression $|2\,\sin\frac{\pi v}{s}|^{-2d}$ is bounded for $v = 1, \ldots, [s/2]$. For $v = 0$, $f(\lambda) \sim |\lambda|^{-2(D+d)}C$, as $\lambda \to 0$ where C is a positive constant.

Figs. 1 and 2 illustrate the behavior of (4) for some values of D, d and s. Fig. 1 shows the symmetric case ($d = 0$). From Fig. 2, we see that there is a strong influence of the fractional

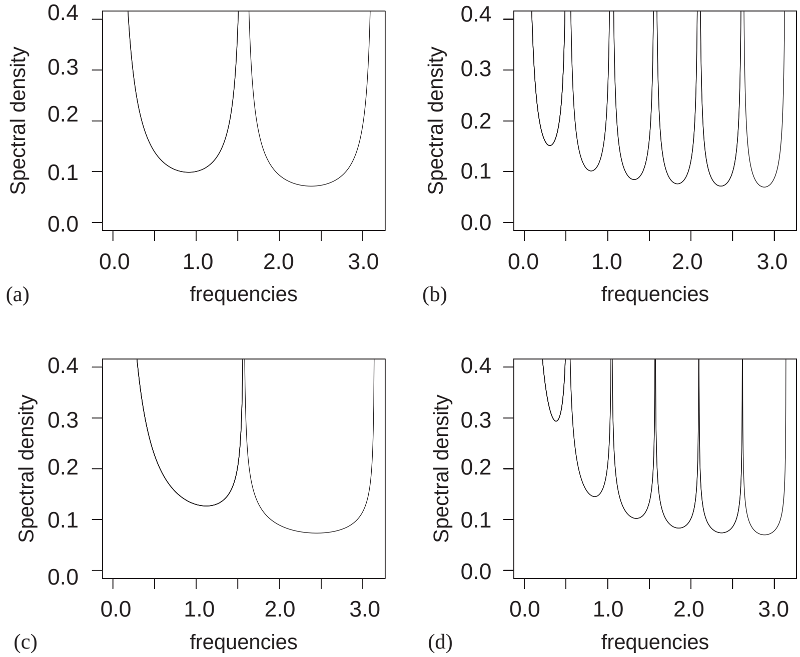

Fig. 2. Spectral densities of model (1): (a) model ARFISMA$(0, 0.2, 0) \times (0, 0.4, 0)_4$; (b) model ARFISMA$(0, 0.2, 0) \times (0, 0.4, 0)_{12}$; (c) model ARFISMA$(0, 0.4, 0) \times (0, 0.2, 0)_4$ and (d) model ARFISMA$(0, 0.4, 0) \times (0, 0.2, 0)_{12}$.

parameter d on the shape of the spectral density, even for those seasonal frequencies far away from the zero frequency.

A more general class of asymmetric seasonal models was considered in Arteche and Robinson (2000). However, for convenience we restrict our attention to the asymmetric and symmetric seasonal cases specified by (1). In Section 3, we discuss several estimation methods for d and D.

3. Estimation methods

3.1. Semiparametric estimates

A multiple linear regression equation is obtained by taking logarithms in expression (4)

$$\log f(\lambda) = \log \frac{\sigma_\varepsilon^2}{2\pi} - D \log[2 \sin(\lambda s/2)]^2 - d \log[2 \sin(\lambda/2)]^2 \tag{6}$$

for $-\pi \leqslant \lambda \leqslant \pi$. Estimates of d and D may be obtained by replacing $f(\lambda)$ by the periodogram $I(\lambda) = n^{-1}|\sum_{t=1}^{n} X_t e^{i\lambda t}|^2$ and then approximating regression (6) by

$$\log I(\lambda) \cong a_0 - D \log[2 \sin(\lambda s/2)]^2 - d \log[2 \sin(\lambda/2)]^2 + U, \tag{7}$$

where a_0 is a constant and $U = \ln \frac{I(\lambda)}{f(\lambda)} - E[\ln \frac{I(\lambda)}{f(\lambda)}]$. This is a natural extension of the non-seasonal method given in Geweke and Porter-Hudak (1983).

Now, assume for simplicity that the number of observations n is divisible by s. We consider the frequencies $\lambda_{v,j} = \frac{2\pi v}{s} + \frac{2\pi j}{n}$, $v = 0, 1, \ldots, [\frac{s}{2}] - 1$, $j = 1, 2, \ldots, m$, for some choice of the bandwidth m which satisfies at least $\frac{1}{m} + \frac{m}{n} \to 0$ as $n \to \infty$. Since λ must lie in $(0, \pi)$, we set $\lambda_{[\frac{s}{2}],j} = \pi - \frac{2\pi j}{n}$. Different estimation methods for D and d may be obtained by appropriate choices of the harmonic frequency λ : GPH$_v$, for $v = 0, 1, \ldots, [s/2]$, is the regression estimator obtained by choosing $\frac{2\pi j}{n}$ frequencies on the right-hand side of the seasonal frequency λ_v with exception at $v = \frac{s}{2}$, as defined previously. For example, if $s = 4$, we obtain the estimators GPH$_0$, GPH$_1$ and GPH$_2$ by using the frequencies $\lambda_{0,j} = \frac{2\pi j}{n}$, $\lambda_{1,j} = \frac{\pi}{2} + \frac{2\pi j}{n}$ and $\lambda_{2,j} = \pi - \frac{2\pi j}{n}$, respectively, for $j = 1, \ldots, m$. In order to avoid overlapping frequencies when estimating d and D using GPH$_v$, we choose m such that $m < \frac{n}{2s}$. Observe that for $m = \frac{n}{2s}$, $\lambda_{v,m}$ corresponds the frequency $\frac{\lambda_v + \lambda_{v+1}}{2}$. Consequently, GPH$_0$ uses frequencies $\lambda_{0,1} = \frac{2\pi}{n}$ to $\lambda_{0,\frac{n}{2s}} = \frac{\pi}{s}$. Note that for large n we have $\lambda_{0,j} \in [0, \frac{\pi}{s}]$. Similarly, for GPH$_1$ $\lambda_{1,j} \in [\frac{\pi}{2}, \frac{\pi}{2} + \frac{\pi}{s}]$ and for GPH$_2$ $\lambda_{2,j} \in [\pi - \frac{\pi}{s}, \pi]$.

GPH$_v$ allows us to obtain estimates of the fractional parameters around each seasonal frequency. Hence, if the estimates of D at each seasonal frequency λ_v are not significantly different, we may assume that the process (1) has $d = 0$, that is, the process has a rigid model representation (Hassler, 1994). Note that GPH$_0$ corresponds to the method proposed by Porter-Hudak (1990) to estimate d and D, that is, taking into consideration only frequencies in a neighborhood of the origin.

The GPH$_T$ method, T for total, calculates the estimates by using all harmonic frequencies in the regression equation (7). On the other hand, the GPH$_P$ method, P for partial, considers a collection of the Fourier frequencies chosen around the right-hand side of the seasonal frequencies (left-hand side when $v = [\frac{s}{2}]$). It should be noted that in the case of $d = 0$, the spectral density around each seasonal frequency is symmetric. Hence, the GPH$_P$ regression is analogous to the regression GPH$_T$ but with fewer observations to be regressed. Consequently, it is expected that GPH$_T$ produces more precise estimates than GPH$_P$ method. Fig. 3 illustrates the region (solid line) where the frequencies are selected from, for each method described above.

3.2. Maximum-likelihood estimates

Assuming that the process $\{X_t\}$ is Gaussian, the log-likelihood function may be expressed as

$$\mathscr{L}_n(\boldsymbol{\theta}) = -\frac{1}{2} \log \det T(f_{\boldsymbol{\theta}}) - \frac{1}{2} \boldsymbol{X}' T(f_{\boldsymbol{\theta}})^{-1} \boldsymbol{X}, \tag{8}$$

where $\boldsymbol{\theta} = (d, D)$ is the parameter vector, $f_{\boldsymbol{\theta}}$ is the spectral density given in (4), $\boldsymbol{X} = (X_1, \ldots, X_n)'$ and T is the variance–covariance matrix of $\{X_t\}$,

$$T_{jk} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f_{\boldsymbol{\theta}}(\lambda) \exp(\mathrm{i}\lambda jk)\, \mathrm{d}\lambda,$$

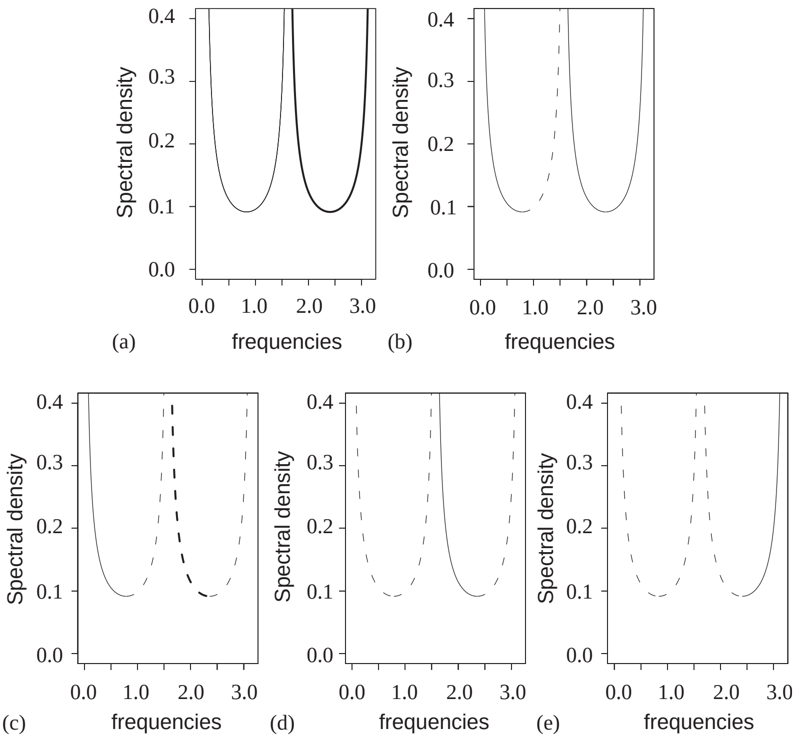

Fig. 3. Examples of the region of the spectral density used in each regression estimator (solid line), $s=4$: (a) region of GPH_T; (b) region of GPH_P; (c) region of GPH_0; (d) region of GPH_1 and (e) region of GPH_2.

see for example Beran (1994, Section 5.3). The ML estimates are obtained by maximizing (8), that is, $\hat{\boldsymbol{\theta}}=\arg\max \mathscr{L}_n(\boldsymbol{\theta})$. The asymptotic variance of $\hat{\boldsymbol{\theta}}$ may be obtained as in Section 5.2 of Taniguchi and Kakizawa (2000), i.e.,

$$\mathrm{var}[\hat{\boldsymbol{\theta}}] \cong \frac{4\pi}{n}\left[\int_{-\pi}^{\pi} \frac{\partial}{\partial \boldsymbol{\theta}} f_{\boldsymbol{\theta}}(\lambda) \frac{\partial}{\partial \boldsymbol{\theta}'} f_{\boldsymbol{\theta}}(\lambda)\,\mathrm{d}\lambda\right]^{-1}.$$

In particular, for an ARFISMA$(0,d,0)\times(0,D,0)_s$, this expression reduces to:

$$\mathrm{var}[\hat{\boldsymbol{\theta}}] \cong \left[n\left(\frac{\pi^4}{36}-a_s^2\right)\right]^{-1}\begin{pmatrix}\dfrac{\pi^2}{6} & -a_s \\ -a_s & \dfrac{\pi^2}{6}\end{pmatrix},$$

where $a_s=\frac{1}{\pi}\int_{-\pi}^{\pi}\{\log|2\sin(\frac{\lambda}{2})|\}\{\log|2\sin(s\frac{\lambda}{2})|\}\,\mathrm{d}\lambda$. For $s=4$, we have

$$\mathrm{var}[\hat{\boldsymbol{\theta}}] \cong \frac{1}{n}\begin{pmatrix} 0.647 & -0.160 \\ -0.160 & 0.647\end{pmatrix}.$$

Hence, $\mathrm{corr}(\hat{d}, \hat{D}) = -0.247$. On the other hand, for $s = 12$, we have

$$\mathrm{var}[\hat{\boldsymbol{\theta}}] \cong \frac{1}{n} \begin{pmatrix} 0.648 & -0.162 \\ -0.162 & 0.648 \end{pmatrix},$$

and then $\mathrm{corr}(\hat{d}, \hat{D}) = -0.250$. The computation of the ML is cumbersome, especially for large sample sizes. In order to make more efficient this method, we used the state space approach proposed by Chan and Palma (1998).

4. Monte Carlo simulation study

In order to assess the finite sample performance of the methods proposed in Section 3, a number of Monte Carlo experiments were carried out. The simulation results give the mean and the MSE of the estimation procedures based on 1000 replications. All calculations were carried out using a Ox program (see for example Doornik, 1999) in a AMD Athlon XP 1800 computer. The performances of the estimates are presented in Tables 1–8. We consider sample sizes $n = 240$, 480 and 3600, and seasonal periods $s = 4$ and 12. The models and the values of D and d are specified in the tables. The best results (smallest bias or MSE) are in boldface. Simulations with other values of d and D—not shown here but available on request—were also considered in the study and the estimators showed similar behavior. The ARFISMA processes were simulated following the method suggested by

Table 1
Estimates of D when $D = 0.2$ and $s = 4$

n	Stat.	Estimators					
		GPH_T	GPH_P	GPH_0	GPH_1	GPH_2	ML
240	Mean	**0.1983**	0.1955	0.1803	0.2023	0.2039	0.1851
	MSE	**0.0061**	0.0078	0.0310	0.0250	0.0208	**0.0038**
480	Mean	**0.1995**	0.1976	0.1864	0.2043	0.2019	0.1962
	MSE	**0.0030**	0.0043	0.0134	0.0131	0.0104	**0.0016**
3600	Mean	**0.1996**	0.1980	0.1961	0.1986	0.1992	0.2063
	MSE	**0.0003**	**0.0003**	0.0011	0.0013	0.0011	**0.0003**

Table 2
Estimates of D when $D = 0.4$ and $s = 4$

n	Stat.	Estimators					
		GPH_T	GPH_P	GPH_0	GPH_1	GPH_2	ML
240	Mean	**0.4147**	**0.4138**	**0.4099**	0.4249	**0.4065**	**0.3902**
	MSE	**0.0074**	0.0103	0.0297	0.0229	0.0254	**0.0039**
480	Mean	**0.4064**	**0.4001**	0.4141	0.4020	0.3841	**0.4040**
	MSE	**0.0026**	0.0034	0.0091	0.0088	0.0133	**0.0019**
3600	Mean	**0.4021**	**0.4015**	**0.4011**	0.4043	**0.3992**	0.4286
	MSE	**0.0003**	**0.0004**	0.0010	0.0012	0.0013	**0.0012**

Table 3
Estimates of D when $D = 0.2$ and $s = 12$

n	Stat.	Estimators				
		GPH_T	GPH_P	GPH_0	GPH_1	GPH_2
240	Mean	**0.2039**	0.2090	0.2128	0.1961	**0.1998**
	MSE	**0.0106**	0.0191	0.1628	0.1065	0.1061
480	Mean	**0.2051**	0.2088	0.2128	0.2466	0.1592
	MSE	**0.0039**	0.0065	0.0443	0.0420	0.0433
3600	Mean	**0.1996**	0.1981	0.2024	0.2040	0.1898
	MSE	**0.0003**	0.0006	0.0028	0.0027	0.0048

n	Stat.	GPH_3	GPH_4	GPH_5	GPH_6	ML
240	Mean	0.1757	**0.1987**	0.2483	0.2315	0.1851
	MSE	0.1648	0.1175	0.1140	0.1574	**0.0061**
480	Mean	0.1943	0.2311	0.1932	0.2249	**0.1972**
	MSE	0.0382	0.0344	0.0672	0.0455	**0.0019**
3600	Mean	0.2015	0.1907	0.2046	0.1940	0.2056
	MSE	0.0042	0.0036	0.0036	0.0051	**0.0002**

Table 4
Estimates of D when $D = 0.4$ and $s = 12$

n	Stat.	Estimators				
		GPH_T	GPH_P	GPH_0	GPH_1	GPH_2
240	Mean	**0.4133**	0.4146	0.3802	0.4930	0.4245
	MSE	**0.0086**	0.0163	0.1557	0.1575	0.1390
80	Mean	**0.4126**	0.4195	**0.3995**	0.4560	0.3965
	MSE	**0.0039**	0.0061	0.0387	0.0384	0.0471
3600	Mean	**0.4006**	**0.3997**	**0.3943**	0.4004	0.3992
	MSE	**0.0003**	**0.0006**	0.0035	0.0037	0.0031

n	Stat.	GPH_3	GPH_4	GPH_5	GPH_6	ML
240	Mean	0.3891	**0.4136**	0.4206	0.3811	0.3854
	MSE	0.1333	0.1081	0.1180	0.1204	**0.0054**
480	Mean	**0.4020**	0.4264	0.4525	**0.4032**	**0.4017**
	MSE	0.0426	0.0506	0.0444	0.0563	**0.0019**
3600	Mean	**0.3978**	0.4068	0.3948	0.4048	0.4234
	MSE	0.0041	0.0036	0.0031	0.0036	**0.0008**

Hosking (1984) with Gaussian noise with unit variance. Tables 1–4 display the results concerned to the model ARFISMA$(0, D, 0)_s$ and the remaining tables give the results related to the ARFISMA$(0, d, 0) \times (0, D, 0)_s$.

From Tables 1 to 4, two of the methods proposed, namely GPH_T and GPH_P, perform very well and much better than the GPH_v. This estimator gives for each v similar estimates and, in practical situations, this may be useful to verify the assumption of a rigid model, that

Table 5
Estimates of d and D when $d=0.1,\ D=0.3$ and $s=4$

n	Stat.	Estimators					
		GPH_T	GPH_P	GPH_0	GPH_1	GPH_2	ML
240	Mean(d)	**0.1086**	**0.1089**	0.1194	—	—	0.0784
	MSE(d)	**0.0051**	**0.0054**	1.8774	—	—	**0.0031**
	Mean(D)	**0.2997**	**0.2981**	0.2998	0.2884	**0.3098**	0.2858
	MSE(D)	**0.0076**	0.0091	2.3461	0.1886	0.0721	**0.0030**
480	Mean(d)	0.1053	0.1052	0.2331	—	—	0.0831
	MSE(d)	**0.0020**	**0.0022**	0.8630	—	—	**0.0019**
	Mean(D)	**0.3032**	**0.3046**	0.1605	0.3147	0.3166	0.2936
	MSE(D)	**0.0030**	**0.0040**	1.0673	0.0480	0.0262	**0.0016**
3600	Mean(d)	**0.0999**	**0.1004**	0.0819	—	—	0.0956
	MSE(d)	**0.0004**	**0.0004**	0.1039	—	—	**0.0002**
	Mean(D)	**0.3013**	**0.3001**	0.3169	0.3053	0.2979	0.3021
	MSE(D)	**0.0003**	0.0004	0.1236	0.0042	0.0023	**0.0002**

Table 6
Estimates of d and D when $d=0.1,\ D=0.3$ and $s=12$

n	Stat.	Estimators			
		GPH_T	GPH_P	GPH_0	GPH_1
240	Mean(d)	**0.1108**	**0.1164**	0.3859	—
	MSE(d)	**0.0075**	0.0095	11.996	—
	Mean(D)	**0.2988**	0.3009	−0.0051	0.4453
	MSE(D)	0.0107	0.0219	16.131	1.1499
480	Mean(d)	**0.1014**	**0.1025**	0.1419	—
	MSE(d)	**0.0023**	**0.0033**	2.9211	—
	Mean(D)	**0.3130**	**0.3077**	0.2718	0.3361
	MSE(D)	**0.0032**	**0.0063**	3.8261	0.2369
3600	Mean(d)	**0.0984**	**0.0980**	0.2167	—
	MSE(d)	**0.00020**	0.0003	0.3618	—
	Mean(D)	**0.3025**	**0.3011**	0.1703	0.2861
	MSE(D)	**0.0003**	0.0006	0.4392	0.0166

n	Stat.	GPH_2	GPH_3	GPH_4	GPH_5	GPH_6	ML
240	Mean(d)	—	—	—	—	—	0.0783
	MSE(d)	—	—	—	—	—	**0.0031**
	Mean(D)	0.0369	0.4335	0.2085	0.2184	0.3353	0.2897
	MSE(D)	0.9057	0.8596	1.0174	1.2959	0.5675	**0.0032**
480	Mean(d)	—	—	—	—	—	0.0806
	MSE(d)	—	—	—	—	—	**0.0016**
	Mean(D)	0.2428	0.2929	0.2661	0.3054	0.2915	0.2949
	MSE(D)	0.3099	0.3045	0.2143	0.3471	0.1038	**0.0014**
3600	Mean(d)	—	—	—	—	—	**0.0957**
	MSE(d)	—	—	—	—	—	**0.0002**
	Mean(D)	0.2908	0.3007	**0.3025**	**0.3002**	0.31372	**0.3028**
	MSE(D)	0.0126	0.0163	0.0146	0.0156	0.0083	**0.0002**

Table 7
Estimates of d and D when $d = 0.2$, $D = 0.4$ and $s = 4$

n	Stat.	Estimators				
		GPH_T	GPH_P	GPH_0	GPH_1	GPH_2
240	Mean(d)	**0.206**	**0.206**	0.153	—	—
	MSE(d)	**0.004**	**0.004**	1.791	—	—
	Mean(D)	**0.404**	**0.402**	0.455	0.473	0.437
	MSE(D)	**0.006**	**0.009**	2.296	0.193	0.061
480	Mean(d)	**0.202**	**0.200**	0.121	—	—
	MSE(d)	**0.002**	**0.003**	0.987	—	—
	Mean(D)	**0.407**	**0.409**	0.492	0.443	0.436
	MSE(D)	**0.003**	**0.004**	1.250	0.074	0.027
3600	Mean(d)	**0.201**	**0.202**	0.179	—	—
	MSE(d)	**0.0002**	**0.0003**	0.117	—	—
	Mean(D)	**0.406**	**0.405**	0.434	0.414	**0.398**
	MSE(D)	**0.0003**	**0.0005**	0.142	0.005	0.003

Table 8
Estimates of d and D when $d = 0.2$, $D = 0.4$ and $s = 12$

n	Stat.	Estimators			
		GPH_T	GPH_P	GPH_0	GPH_1
240	Mean(d)	**0.208**	0.211	0.657	—
	MSE(d)	**0.006**	0.008	10.040	—
	Mean(D)	**0.414**	**0.410**	−0.062	0.394
	MSE(D)	**0.010**	0.017	13.850	0.995
480	Mean(d)	**0.204**	**0.203**	0.242	—
	MSE(d)	**0.003**	**0.004**	3.105	—
	Mean(D)	**0.403**	**0.406**	0.337	0.421
	MSE(D)	**0.003**	**0.005**	4.112	0.225
3600	Mean(d)	**0.205**	**0.205**	0.133	—
	MSE(d)	**0.0003**	**0.0003**	0.374	—
	Mean(D)	**0.405**	**0.405**	0.494	**0.405**
	MSE(D)	**0.0003**	**0.0006**	0.485	0.014

n	Stat.	GPH_2	GPH_3	GPH_4	GPH_5	GPH_6
240	Mean(d)	—	—	—	—	—
	MSE(d)	—	—	—	—	—
	Mean(D)	0.418	0.483	0.377	0.341	0.379
	MSE(D)	0.835	1.093	1.412	1.003	0.437
480	Mean(d)	—	—	—	—	—
	MSE(d)	—	—	—	—	—
	Mean(D)	0.447	0.430	0.441	**0.408**	**0.409**
	MSE(D)	0.185	0.214	0.303	0.327	0.104
3600	Mean(d)	—	—	—	—	—
	MSE(d)	—	—	—	—	—
	Mean(D)	0.415	0.421	0.394	**0.409**	**0.397**
	MSE(D)	0.015	0.015	0.017	0.019	0.009

is, $d=0.0$. The good behavior of these methods is directly related to the fact that the spectral density is symmetric around all seasonal frequencies (see Fig. 1 in the previous section). In the regression methods, better estimates are obtained from GPH_T (smaller bias and MSE) since this method involves all harmonic frequencies in the regression equation. For small sample sizes, the MSE of all methods increases with s due to the fact that fewer Fourier frequencies are involved in the regression equation (compare for example Tables 1 and 2 for $n = 240$). This effect becomes insignificant as n increases. Note that the approximate parameter variance based on Gaussian processes with $s = 4$ are 0.0027, 0.0014 and 0.0002 for $n = 240, 480$ and 3600, respectively. For $s = 12$ the parameter variances are 0.0025, 0.0013 and 0.0002 for $n = 240, 480$ and 3600, respectively. Thus, the proposed estimators GPH_T and GPH_P have asymptotic variances close to the optimal values for $n = 3600$. Observe that the ML estimates always give smaller MSE than the regression methods. However, the bias of this estimator is, in general, larger then one produced by GPH_T. For large n, GPH_T, GPH_P and ML estimators have similar MSE. It should be noted that the ML estimator required a lot of computational time and in some cases the simulations stopped before reaching the 1000 replications for $n = 3600$.

Tables 5–8 present the estimation results when the complete model (1) is considered in the study for stationary and non-stationary cases.

Observe that overall, estimators GPH_T and GPH_P have better performance than the others regression methods, in terms of bias and MSE. It should be noted that the estimators GPH_v, for $v=1, \ldots, [\frac{s}{2}]$, are only used for estimating D since its seasonal frequency is away from zero. For comparison purposes, we also include the estimates produced by the GPH_0 method suggested by Porter-Hudak (1990). However, as noted by Porter-Hudak, it should be expected that this estimator combines the effect of d and D, since these parameters cannot be identified separately. As shown in the tables, this estimator produces biased estimates of both long-memory parameters. However, as v increases the bias and the mean squared error of the estimate of D decreases. Thus, by combining GPH_0 and $\mathrm{GPH}_{[s/2]}$ the value of the both parameters may be identified, where $\mathrm{GPH}_{[s/2]}$ provides the estimate of D and $\mathrm{GPH}_0 - \mathrm{GPH}_{[s/2]}$ can be used to estimate d. For example, in Table 7 for $n=3600$ we have from GPH_0 that $\hat{d} + \hat{D} = 0.613$ and from $\mathrm{GPH}_{[s/2]}$ we have $\hat{D} = 0.398$. Consequently, an estimate for d is $\hat{d} = 0.215$.

As previously mentioned, the ML always gives smaller MSE. However, for large n the MSE of the regression methods GPH_T and GPH_P are very close to the ML method. It should be noted that the ML here is restricted to the stationary case. As far as the authors know, the asymptotic properties of the ML in the non-stationary case are not established yet. Hence, this method is not considered in the Tables 7 and 8 where the results of the non-stationary simulation studies are presented.

The following figures are examples to illustrate empirically the linear relation between the estimates of d and D using the regression methods and the ML estimator when $(s=4)$, $d=0.1$ and $D=0.3$. Fig. 4 gives the scatterplot of the GPH_T and GPH_v $(v=0, 1, 2)$ estimates of d and D and the sample autocorrelation. This example illustrates that the estimates of d and D are highly correlated when using the GPH_v method. Fig. 5 gives the scatterplot of the ML estimates of d and D. The theoretical and sample autocorrelations between the estimates are -0.25 and -0.279, respectively. The sample correlation of the GPH_T estimates is very

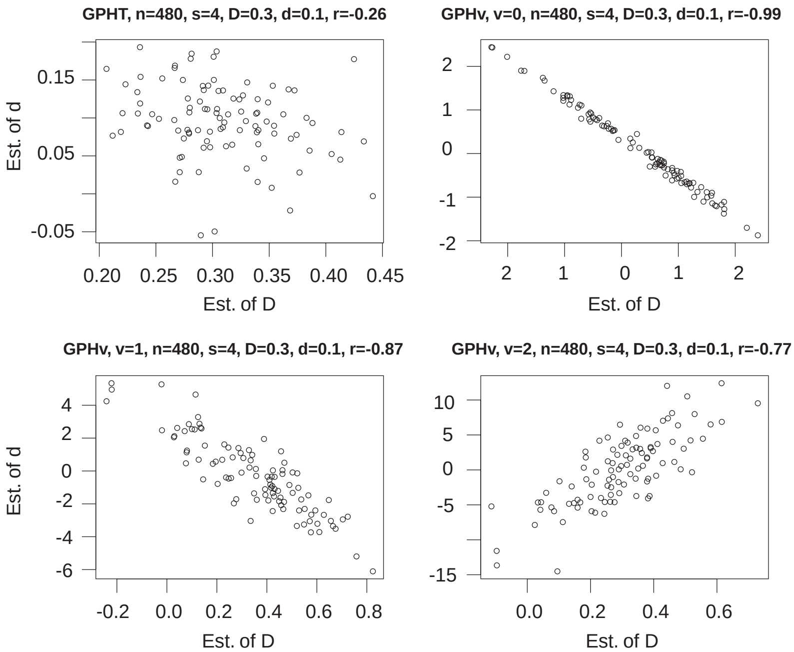

Fig. 4. Scatterplot of the GPH_T and GPH_v estimates, $d = 0.1$ and $D = 0.3$ for $s = 4$ and $n = 480$ based on 1000 replications.

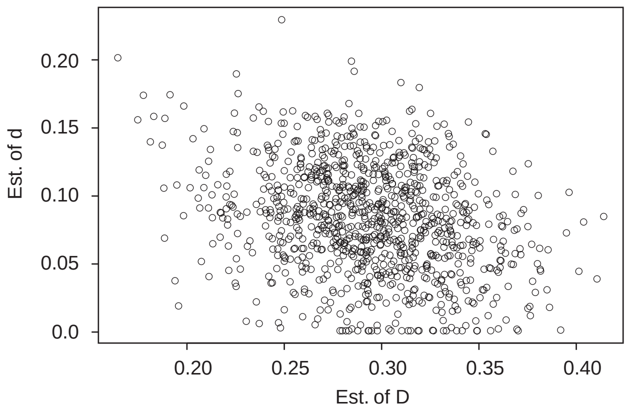

Fig. 5. Scatterplot of the ML estimates of $d = 0.1$ and $D = 0.3$ for $s = 4$ and $n = 480$ based on 1000 replications.

close to the one given by the ML estimates. For $v = 2$, the sample correlation is positive due to the fact that the regression estimates are obtained using the frequencies of the left-side of the frequency $\lambda = \pi$.

5. Conclusions

As suggested by the Monte Carlo experiments, two of the estimation methodologies proposed in this paper namely, GPH_T and GPH_P, seem to perform very well even for small sample sizes. Consequently, they are reasonable methods to deal with seasonal long-range dependent data. The methods considered are natural extensions of the one proposed originally by Geweke and Porter-Hudak (1983) for non-seasonal long-memory processes. The approach proposed by Porter-Hudak (1990) was also considered in our study, but it showed poor performance. In addition, the methods involving most of the frequencies yield better estimates than those regression techniques based only on neighborhoods of the seasonal frequencies. Also, for large n, the proposed methods are very competitive when compared to the maximum-likelihood estimator. Since the results from the simulations are very encouraging, it would be interesting to study the performance of these estimators for more general seasonal models and to extend them to the multivariate case.

Acknowledgements

The first two authors gratefully acknowledge partial financial support from PIBIC-CNPq/ Brazil. Wilfredo Palma acknowledges partial financial support from Fondecyt Grants 1040934/1030588. The authors wish to thank the editor and the two anonymous referees for their comments on the earlier version of this paper. Some results of the present paper were obtained when V.A. Reisen was a visitor at UMIST, Manchester, UK, and at the University of Padua, Italy. He thanks Prof. T.S. Rao (UMIST) and Prof. S. Bordignon (Italy) for the invitation.

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