



On the eigenstructure of generalized fractional processes

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Abstract

This work establishes bounds for the eigenvalues of the covariance matrix from a general class of stationary processes. These results are applied to the statistical analysis of the large sample behavior of estimates and testing procedures of generalized long memory models, including Seasonal ARFIMA and k -factor GARMA processes, among others.

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1. Introduction

Time series data exhibiting both long memory and cyclical behavior have been extensively documented during the last decade. For example, these features are present in monetary aggregates (Porter-Hudak, 1990), revenue data (Ray, 1993), inflation rates (Hassler and Wolters, 1995) and monthly flows of the Nile River (Montanari et al., 2000), among others. To account for this behavior, a number of models have been proposed. For instance, Porter-Hudak (1990) introduces the seasonal autoregressive fractionally integrated moving average (SARFIMA) process and Woodward et al. (1998) propose the k -factor Gegenbauer autoregressive moving average (k -factor GARMA) process. A SARFIMA process (X_t) with period s satisfies the difference equation

$$\phi(B)(1 - B)^d \Phi(B^s)(1 - B^s)^{d_1} (X_t - \mu) = \theta(B)\Theta(B^s)\varepsilon_t,$$

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where B is the backward shift operator, $s \geq 2$ is an integer, (ε_t) is an uncorrelated zero mean sequence with variance σ_ε^2 , $\phi, \Phi, \theta, \Theta$ are polynomials such that ϕ and Φ have no zeroes on the unit circle, and $d + d_1 < 1/2$, $d_1 < 1/2$. Under these assumptions, (X_t) is stationary with mean μ and its spectral measure is absolutely continuous with respect to $d\nu$, the Lebesgue measure on $I = [-\pi, \pi]$. The corresponding spectral density f is given by

$$f(\nu) = \frac{\sigma_\varepsilon^2}{2\pi} \left| \frac{\theta(e^{i\nu})\Theta(e^{is\nu})}{\phi(e^{i\nu})\Phi(e^{is\nu})} \right|^2 \prod_{j=0}^k |(e^{i2\pi j/s} - e^{i\nu})(e^{-i2\pi j/s} - e^{i\nu})|^{-2d_j}, \tag{1}$$

where $k = [s/2]$, $d_0 = (d + d_1)/2$, $d_j = d_1$ for $1 \leq j \leq k - 1$, $d_k = d_1$ if s is odd, and $d_k = d_1/2$ if s is even. A k -factor GARMA process (X_t) satisfies the difference equation

$$\phi(B) \prod_{j=1}^k (1 - 2B \cos \nu_j + B^2)^{d_j} (X_t - \mu) = \theta(B)\varepsilon_t,$$

where ϕ has no zeroes on the unit circle, the ν_j are distinct values in $[0, \pi]$, all $d_j \neq 0$, $d_j < 1/2$ whenever $\nu_j \neq 0$ and $\nu_j \neq \pi$, and $d_j < 1/4$ when $\nu_j = 0$ or $\nu_j = \pi$. (X_t) is stationary with mean μ and its spectral measure is absolutely continuous with respect to $d\nu$ with density f given by

$$f(\nu) = \frac{\sigma_\varepsilon^2}{2\pi} \left| \frac{\theta(e^{i\nu})}{\phi(e^{i\nu})} \right|^2 \prod_{j=1}^k |(e^{i\nu_j} - e^{i\nu})(e^{-i\nu_j} - e^{i\nu})|^{-2d_j}. \tag{2}$$

The long memory property of SARFIMA and k -factor GARMA processes is reflected by the peculiar shape of their spectral density around the frequencies ν_j with $\nu_j = 2\pi j/s$ for a SARFIMA process. Indeed, in (1) and (2), $f(\nu)$ behaves like $|\nu - \nu_j|^{-2d_j}$ as $\nu \rightarrow \nu_j$. Thus, when $0 < d_j < 1/2$, $f(\nu)$ is unbounded at $\nu = \nu_j$.

In this paper, we consider a (possibly complex-valued) stationary process (X_t) with absolutely continuous spectral measure whose derivative f satisfies (5) where g is bounded away from zero and infinity, the ν_j are arbitrary distinct values in $(-\pi, \pi)$ and all $d_j < 1/2$. We establish some bounds for the eigenvalues $\lambda_{1,n}, \dots, \lambda_{n,n}$ of the covariance matrix $\Gamma_n = (\gamma_{i-j})_{i,j=1}^n$ where $\gamma_j = \int_I e^{ij\nu} f(\nu) d\nu$ and n is an arbitrary positive integer. Since Γ_n is positive semidefinite, it is Hermitian and all $\lambda_{k,n}$ are nonnegative. By using the interlacing eigenvalues theorem for bordered Hermitian matrices (see [Horn and Johnson, 1991](#), Theorem 4.3.8), one easily obtain that for any fixed integer $k \geq 1$, the sequences $(\lambda_{k,n})_{n \geq k}$ and $(\lambda_{n+1-k,n})_{n \geq k}$ are respectively nonincreasing and nondecreasing. Moreover, if we denote by m and M the essential infimum and supremum of f , respectively, it was shown by [Grenander and Szegö \(1958, Chapter 5\)](#) that $2\pi m \leq \lambda_{k,n} \leq 2\pi M$, and that for any fixed $k \geq 1$,

$$\lim_{n \rightarrow \infty} \lambda_{k,n} = 2\pi m, \tag{3}$$

$$\lim_{n \rightarrow \infty} \lambda_{n+1-k,n} = 2\pi M. \tag{4}$$

The limit relation (4) holds also in case $M = +\infty$. The same authors established the following result about the order of magnitude of the differences $\lambda_{k,n} - 2\pi m$ and $2\pi M - \lambda_{n+1-k,n}$ as n becomes large.

Theorem 1 (Grenander and Szegő, 1958). *Under the following assumptions:*

- (i) *f is continuous on the compact set I and $f(-\pi) = f(\pi)$;*
- (ii) *the infimum of f is attained at only one value $v = v_0 \pmod{2\pi}$;*
- (iii) *f has a continuous second derivative in a certain neighborhood of v_0 and $f''(v_0) \neq 0$;*

we have for any fixed $k \geq 1$, $\lambda_{k,n} - 2\pi m \sim C/n^2$ as $n \rightarrow \infty$ where C is a positive constant. A similar result holds for $2\pi M - \lambda_{n+1-k,n}$.

In Theorem 2, we consider a spectral density of special type which is not necessarily continuous nor bounded on I and we establish some bounds for the eigenvalues $\lambda_{k,n}$. In Corollary 1, we consider the case where $m = 0$ and $M = +\infty$ and we establish some bounds on the rate of convergence of $\lambda_{k,n}$ to zero, and $\lambda_{n+1-k,n}$ to $+\infty$, respectively, as $n \rightarrow \infty$ and k is fixed. As applications, we show that the bounds established in Theorem 2 give valuable clues about the asymptotic behavior of the best linear unbiased estimator (BLUE) of the mean of (X_t) , the sample mean, and the consistency of the time series linear discriminant functions (LDF).

This paper is organized as follows. The results on the behavior of the eigenvalues of Γ_n are presented in Section 2. It starts with a preliminary result, and then proceeds with the main result contained in Theorem 2. Applications of this theorem to the estimation of the location of the process (X_t) and to the analysis of the consistency of the LDF are discussed in Section 3.

2. Main result

Hereafter, for any set S, we denote by $\mathbb{1}_S$ the indicator function of S.

Lemma 1. *Let $I = (a, b)$ be an interval in \mathbb{R} , H the set of functions mapping I into \mathbb{R} , c, d positive real numbers satisfying $c(b - a) \geq d$, f a monotone integrable function in H, $P = \{h \in H; 0 \leq h \leq c, \int_I h = d\}$, and $\alpha = \inf_{h \in P} \int_I fh$, $\beta = \sup_{h \in P} \int_I fh$. Then, if f is nonincreasing (resp. nondecreasing), we have α (resp. β) = $c \int_I f \mathbb{1}_{(b-d/c, b)}$, β (resp. α) = $c \int_I f \mathbb{1}_{(a, a+d/c)}$.*

Proof. We show that f nonincreasing implies $\alpha = c \int_I f \mathbb{1}_{(b-d/c, b)}$. The other results are obtained similarly. Let $I_1 = (a, b - d/c)$, $I_2 = (b - d/c, b)$, $h_0 = c \mathbb{1}_{I_2}$, and $h \in P$. Since $c(b - a) \geq d$, $h_0 \in P$. We have $\int_I f(h - h_0) = \int_{I_1} fh + \int_{I_2} f(h - c)$. Now

$$\int_{I_1} fh \geq f(b - d/c) \int_{I_1} h = f(b - d/c) \int_{I_2} (c - h).$$

Therefore,

$$\int_I f(h - h_0) \geq \int_{I_2} (f(b - d/c) - f)(c - h) \geq 0,$$

which means that $\int_I fh_0 = \min_{h \in P} \int_I fh$. \square

The following theorem applies to a large class of long memory models including both the SARFIMA and the k-factor GARMA.

Theorem 2. Let (X_t) be a stationary process with absolutely continuous spectral measure whose derivative f with respect to the Lebesgue measure on $I = [-\pi, \pi]$ satisfies

$$f(v) = g(v) \prod_{j=1}^m |e^{iv_j} - e^{iv}|^{-2d_j} \tag{5}$$

where function g complies with the boundedness condition $0 < \alpha \leq g \leq \beta < \infty$ for some constants α, β , the v_j are distinct values in $(-\pi, \pi)$, all $d_j \neq 0$ and $d_1 \leq \dots \leq d_m < 1/2$. Let $\lambda_{1,n} \leq \dots \leq \lambda_{n,n}$ be the eigenvalues of the covariance matrix $\Gamma_n = (\gamma_{i-j})_{i,j=1}^n$ where $\gamma_j = \int_I e^{ijv} f(v) dv$. Then there exist positive constants C_1, C_2 such that

$$C_1 n^{2\min(0, d_1)} \leq \lambda_{k,n} \leq C_2 n^{2\max(0, d_m)} \tag{6}$$

for all $n \geq 1$ and for all $k \in \{1, \dots, n\}$.

Proof. Let $f_j(v) = |e^{iv_j} - e^{iv}|^{-2d_j}$. If $d_j > 0$, f_j is positive, has a pole at $v = v_j$ and is continuous on $I \setminus \{v_j\}$; thus $f_j \geq a_j > 0$ for some constant a_j . If $d_j < 0$, f_j has a zero at $v = v_j$ and is continuous on I ; thus $f_j \leq b_j < \infty$ for some constant b_j . Therefore, we deduce from (5) that $f \geq a > 0$ if $d_1 > 0$, and $f \leq b < \infty$ if $d_m < 0$, for some constants a, b . Assume that $d_1 < 0$ and $d_m > 0$, and let k be the integer such that $d_k < 0 < d_{k+1}$. We have

$$\begin{aligned} a'' \prod_{j=1}^k |e^{iv_j} - e^{iv}|^{-2d_1} &\leq a' \prod_{j=1}^k |e^{iv_j} - e^{iv}|^{-2d_j} \leq f(v) \\ &\leq b' \prod_{j=k+1}^m |e^{iv_j} - e^{iv}|^{-2d_j} \leq b'' \prod_{j=k+1}^m |e^{iv_j} - e^{iv}|^{-2d_m} \end{aligned}$$

for some positive constants a'', a', b', b'' . Now,

$$\forall v \in I, \quad 0 < \frac{\sin\left(\frac{\pi+|v_j|}{2}\right)}{\frac{\pi+|v_j|}{2}} \leq \left| \frac{\sin\left(\frac{v-v_j}{2}\right)}{\frac{v-v_j}{2}} \right| = \left| \frac{e^{iv} - e^{iv_j}}{v - v_j} \right| \leq 1. \tag{7}$$

Therefore, there exist positive constants a, b such that

$$af \leq f \leq b\bar{f}, \tag{8}$$

where

$$\underline{f}(v) = \prod_{j=1}^k |v - v_j|^{-2d_1}, \tag{9}$$

$$\bar{f}(v) = \prod_{j=k+1}^m |v - v_j|^{-2d_m}. \tag{10}$$

We are going to show that there exist $C_1 > 0$ and $C_2 > 0$ such that $\lambda_{1,n} \geq C_1 n^{2\min(0, d_1)}$ and $\lambda_{n,n} \leq C_2 n^{2\max(0, d_m)}$ for all $n \geq 1$. Let $n \geq 1$ and $\mathcal{C} = \{x \in \mathbb{C}^n; x^*x = 1\}$ where x^* is the transpose

conjugate of x . Since Γ_n is Hermitian, we have $\lambda_{1,n} = \min_{x \in \mathcal{C}} x^* \Gamma_n x$ and $\lambda_{n,n} = \max_{x \in \mathcal{C}} x^* \Gamma_n x$. Let $x = (x_1, \dots, x_n)' \in \mathcal{C}$, we have

$$x^* \Gamma_n x = \sum_{k,l=1}^n x_k^* \gamma_{k-l} x_l = \int_I \left| \sum_{k=1}^n x_k e^{-ikv} \right|^2 f(v) dv. \tag{11}$$

If $d_1 > 0$, $f \geq a > 0$ and we deduce from (11) that $x^* \Gamma_n x \geq 2\pi a$, so that $\lambda_{1,n} \geq 2\pi a$. Similarly, if $d_m < 0$, $f \leq b \leq \infty$ and we get that $\lambda_{n,n} \leq 2\pi b$. In what follows, we study the case $d_1 < 0$ and $d_m > 0$. Let H be the set of functions mapping I into \mathbb{R} and $P_n = \{h \in H; 0 \leq h \leq n, \int_I h = 2\pi\}$. The function $v \mapsto |\sum_{k=1}^n x_k e^{-ikv}|^2$ belongs to P_n . Therefore, we deduce from (11) that $\lambda_{1,n} \geq \inf_{h \in P_n} \int_I h f$ and $\lambda_{n,n} \leq \sup_{h \in P_n} \int_I h f$. Since any $h \in P_n$ is nonnegative, it results from (8) that

$$\lambda_{1,n} \geq a \inf_{h \in P_n} \int_I h f, \tag{12}$$

$$\lambda_{n,n} \leq b \sup_{h \in P_n} \int_I h \bar{f}. \tag{13}$$

Let us first study $\inf_{h \in P_n} \int_I h f$. We arrange the points v_1, \dots, v_k in ascending order, $v_{j_1} < \dots < v_{j_k}$. According to (9), f has k zeroes at v_{j_1}, \dots, v_{j_k} and $k-1$ maxima at ξ_1, \dots, ξ_{k-1} such that $v_{j_i} < \xi_i < v_{j_{i+1}}$ for all $i \in \{1, \dots, k-1\}$. f is nonincreasing on $I_1 = [-\pi, v_{j_1})$; for all $i \in \{1, \dots, k-1\}$, f is nondecreasing on $I_{2i} = (v_{j_i}, \xi_i)$ and f is nonincreasing on $I_{2i+1} = (\xi_i, v_{j_{i+1}})$; and f is nondecreasing on $I_{2k} = (v_{j_k}, \pi]$. Since f can be written almost everywhere as $f = \sum_{i=1}^{2k} \mathbb{1}_{I_i} f$, we have

$$\inf_{h \in P_n} \int_I h f = \inf_{h \in P_n} \sum_{i=1}^{2k} \int_{I_i} h f \geq \sum_{i=1}^{2k} \inf_{h \in P_n} \int_{I_i} h f. \tag{14}$$

For all $i \in \{1, \dots, 2k\}$, I_i is nonempty. There exists an integer N such that for all $n \geq N$, $n v(I_i) \geq 2\pi$ for each i . Get $n \geq N$, we deduce from Lemma 1 that

$$\forall i \in \{0, \dots, k-1\}, \inf_{h \in P_n} \int_{I_{2i+1}} h f = n \int_{I_{2i+1}} \mathbb{1}_{(v_{j_{i+1}} - 2\pi/n, v_{j_{i+1}})} f, \tag{15}$$

$$\forall i \in \{1, \dots, k\}, \inf_{h \in P_n} \int_{I_{2i}} h f = n \int_{I_{2i}} \mathbb{1}_{(v_{j_i}, v_{j_i} + 2\pi/n)} f. \tag{16}$$

Now,

$$\forall v \in I_{2i+1}, \ln f(v) \geq -2d_1 \ln |v - v_{j_{i+1}}| - 2d_1 \sum_{l=1}^i \ln |\xi_l - v_{j_l}| - 2d_1 \sum_{l=i+2}^k \ln |v_{j_{i+1}} - v_{j_l}|,$$

$$\forall v \in I_{2i}, \ln f(v) \geq -2d_1 \ln |v - v_{j_i}| - 2d_1 \sum_{l=1}^{i-1} \ln |v_{j_i} - v_{j_l}| - 2d_1 \sum_{l=i+1}^k \ln |\xi_l - v_{j_l}|.$$

Hence, there exist positive constants c_1, \dots, c_{2k} such that

$$\forall v \in I_{2i+1}, f(v) \geq c_{2i+1} |v - v_{j_{i+1}}|^{-2d_1}, \tag{17}$$

$$\forall v \in I_{2i}, f(v) \geq c_{2i} |v - v_{j_i}|^{-2d_1}. \tag{18}$$

Combining (15) and (17), we get

$$\inf_{h \in P_n} \int_{I_{2i+1}} h \underline{f} \geq nc_{2i+1} \int_{v_{j_{i+1}} - 2\pi/n}^{v_{j_{i+1}}} |v - v_{j_{i+1}}|^{-2d_1} dv = \frac{c_{2i+1}(2\pi)^{1-2d_1}}{1 - 2d_1} n^{2d_1} \tag{19}$$

for all $i \in \{0, \dots, k - 1\}$, and using (16) and (18), we obtain

$$\inf_{h \in P_n} \int_{I_{2i}} h \underline{f} \geq nc_{2i} \int_{v_{j_i}}^{v_{j_i} + 2\pi/n} |v - v_{j_i}|^{-2d_1} dv = \frac{c_{2i}(2\pi)^{1-2d_1}}{1 - 2d_1} n^{2d_1} \tag{20}$$

for all $i \in \{1, \dots, k\}$. Finally, we deduce from (12), (14), (19) and (20), that there exists $C > 0$ such that $\lambda_{1,n} \geq Cn^{2d_1}$ for all $n \geq N$, which is equivalent to the existence of a positive constant C_1 such that $\lambda_{1,n} \geq C_1 n^{2d_1}$ for all $n \geq 1$.

Let us now study $\sup_{h \in P_n} \int_I h \bar{f}$. We arrange the points v_{k+1}, \dots, v_m in ascending order, $v_{j_{k+1}} < \dots < v_{j_m}$. According to (10), \bar{f} has $m - k$ poles at $v_{j_{k+1}}, \dots, v_{j_m}$ and $m - k - 1$ minima at $\eta_1, \dots, \eta_{m-k-1}$ such that $v_{j_{k+i}} < \eta_i < v_{j_{k+i+1}}$ for all $i \in \{1, \dots, m - k - 1\}$. \bar{f} is nondecreasing on $I_1 = [-\pi, v_{j_{k+1}})$; for all $i \in \{1, \dots, m - k - 1\}$, \bar{f} is nonincreasing on $I_{2i} = (v_{j_{k+i}}, \eta_i)$ and \bar{f} is nondecreasing on $I_{2i+1} = (\eta_i, v_{j_{k+i+1}})$; and \bar{f} is nonincreasing on $I_{2(m-k)} = (v_{j_m}, \pi]$. The counterpart of (14) is

$$\sup_{h \in P_n} \int_I h \bar{f} \leq \sum_{i=1}^{2(m-k)} \sup_{h \in P_n} \int_{I_i} h \bar{f}, \tag{21}$$

and we deduce from Lemma 1 that for all $n \geq N$, we have

$$\forall i \in \{0, \dots, m - k - 1\}, \sup_{h \in P_n} \int_{I_{2i+1}} h \bar{f} = n \int_{I_{2i+1}} \mathbb{1}_{(v_{j_{k+i+1}} - 2\pi/n, v_{j_{k+i+1}})} \bar{f},$$

$$\forall i \in \{1, \dots, m - k\}, \sup_{h \in P_n} \int_{I_{2i}} h \bar{f} = n \int_{I_{2i}} \mathbb{1}_{(v_{j_{k+i}}, v_{j_{k+i}} + 2\pi/n)} \bar{f}.$$

The counterpart of Eqs. (17) and (18) are

$$\forall v \in I_{2i+1}, \bar{f}(v) \leq c_{2i+1} |v - v_{j_{k+i+1}}|^{-2d_m},$$

$$\forall v \in I_{2i}, \bar{f}(v) \leq c_{2i} |v - v_{j_{k+i}}|^{-2d_m},$$

and using that $d_m < 1/2$, easy calculations show that there exists a positive constant c such that $\sup_{h \in P_n} \int_{I_i} h \bar{f} \leq cn^{2d_m}$ for all $i \in \{1, \dots, 2(m - k)\}$ and for all $n \geq N$. Finally, using (13) and (21), we obtain that there exists a positive constant C_2 such that $\lambda_{n,n} \leq C_2 n^{2d_m}$ for all $n \geq 1$. \square

The following result is an immediate consequence of Theorem 2, (3) and (4).

Corollary 1. Assume that $d_1 < 0$ and $d_m > 0$ in Theorem 2, then for any fixed $k \geq 1$,

$$\lambda_{k,n} \rightarrow 0 \text{ and } 1/\lambda_{k,n} = O(n^{-2d_1}),$$

$$\lambda_{n+1-k,n} \rightarrow +\infty \text{ and } \lambda_{n+1-k,n} = O(n^{2d_m}),$$

as $n \rightarrow \infty$.

Example 1. Let $f(v) = |e^{iv_0} - e^{iv}|^{-2d} = (2 \sin \frac{v-v_0}{2})^{-2d}$ where $v_0 \in (-\pi, \pi)$ and $d < 0$. Then f satisfies assumptions (i) and (ii) of Theorem 1, and complies with assumption (iii) only when $d = -1$. It results from Theorem 1 and Corollary 1 that for any fixed $k \geq 1$, $\lambda_{k,n} \sim C/n^2$ when $d = -1$ and $1/\lambda_{k,n} = O(n^{-2d})$ for any $d < 0$, as $n \rightarrow \infty$.

3. Applications

Theorem 2 provides informations about the asymptotic behavior of estimates and testing procedures for generalized fractional processes. Two applications are given below.

Mean estimation. Let (X_t) be a stationary process with mean $\mu = E(X_t)$ and covariance function (γ_j) with $\gamma_0 > 0$. The spectral measure of (X_t) is assumed to be absolutely continuous, so that $\gamma_j = \int_I e^{ijv} f(v) dv$. Since $\gamma_0 > 0$, the covariance matrix $\Gamma_n = (\gamma_{i-j})_{i,j=1}^n$ is invertible for all $n \geq 1$. Consider the problem of estimating μ as a linear combination of X_1, \dots, X_n . The most popular unbiased estimator of μ is the sample mean $\bar{X}_n = n^{-1} \sum_{i=1}^n X_i$, and the BLUE is $\hat{\mu}_n = (u' \Gamma_n^{-1} u)^{-1} u' \Gamma_n^{-1} X$ where $u = (1, \dots, 1)'$ and $X = (X_1, \dots, X_n)'$. We have $\text{Var}(\bar{X}_n) = n^{-2} u' \Gamma_n u$ and $\text{Var}(\hat{\mu}_n) = (u' \Gamma_n^{-1} u)^{-1}$. If f is piecewise continuous, with no discontinuities at $v=0$, and $0 < f < \infty$, then both \bar{X}_n and $\hat{\mu}_n$ have asymptotic variance $2\pi f(0)/n$ as $n \rightarrow \infty$ (see Grenander and Rosenblatt, 1957, Section 7.3). Consider now the case when f diverges at $v=0$. Let $f_d(v) = |1 - e^{iv}|^{-2d}$ with $0 < d < 1/2$, and suppose that function g has the form

$$g(v) = h(v) \prod_{j=1}^k |v - v_j|^{-2d_j}, \tag{22}$$

where the v_j are distinct values in I , all d_j are negative, and h is positive, bounded away from zero, integrable over I and continuous at $v=0$. It was shown by Adenstedt (1974, Theorem 6.1) that if $f(v) = f_d(v)g(v)$ with $g(0) > 0$, then $\text{Var}(\hat{\mu}_n) \sim Cn^{2d-1}$ as $n \rightarrow \infty$. On the other hand, if $0 < d < 1/2$ and $f(v) = |v|^{-2d}b(v)$ where b is of bounded variation and slowly varying at 0 in the sense of Zygmund, then $\gamma_n \sim Cn^{2d-1}$ (see Zygmund, 1959, Section V, Theorem 2.24), and this implies that $\text{Var}(\bar{X}_n) \sim C'n^{2d-1}$ (see Samarov and Taqqu, 1988, Theorem 2). Now, assume that $f(v) = f_d(v)g(v)$ with $0 < d < 1/2$ and g is given by (22) where the v_j are distinct nonzero values in $(-\pi, \pi)$, all d_j are negative and h is positive and bounded away from zero and infinity. We deduce from (7) that f satisfies the assumptions in Theorem 2, and thus $\lambda_{n,n} \leq C_2 n^{2d}$ for all $n \geq 1$. Therefore, $\text{Var}(\bar{X}_n) \leq \lambda_{n,n}/n \leq C_2 n^{2d-1}$ and $\text{Var}(\hat{\mu}_n) \leq \lambda_{n,n}/n \leq C_2 n^{2d-1}$ for all $n \geq 1$. Compared to Adenstedt (1974, Theorem 6.1), our result is weaker but does not assume the continuity of h at $v=0$. Similarly, the bound for $\text{Var}(\bar{X}_n)$ does not require the bounded variation and slowly varying condition.

Example 2. Let (X_t) be an ARFIMA process satisfying the difference equation, $\phi(B)(1 - B)^d(X_t - \mu) = \theta(B)\varepsilon_t$ where ϕ and θ have no zeroes on the unit circle. If $-1/2 < d < 0$, then $\lambda_{1,n} \asymp n^{2d}$, and if $0 < d < 1/2$, then $\lambda_{n,n} \asymp n^{2d}$, where $u_n \asymp v_n$ means that there exist some positive constants C_1, C_2 such that $C_1 \leq u_n/v_n \leq C_2$ for all $n \geq 1$. This can be proved as follows. Suppose that $-1/2 < d < 1/2$, $d \neq 0$. Under the assumption that ϕ and θ have all their zeroes outside the unit disk, it was shown by Brockwell and Davis (1991, Theorem 13.2.2) that $\gamma_n \sim Cn^{2d-1}$ as $n \rightarrow \infty$ where $C \neq 0$. In fact, this result can be established similarly under the weakest assumption that ϕ

and θ have no zeroes on the unit circle. Since $\gamma_n \sim Cn^{2d-1}$, we deduce from the proof of Theorem 2 of Samarov and Taquq (1988) that $\text{Var}(\bar{X}_n) \sim C'n^{2d-1}$ where $C' > 0$. Now,

$$\lambda_{1,n} \leq n\text{Var}(\bar{X}_n) \leq \lambda_{n,n}.$$

Hence, there exist some positive constants C_1, C_2 such that $\lambda_{1,n} \leq C_2n^{2d}$ and $\lambda_{n,n} \geq C_1n^{2d}$ for all $n \geq 1$. Combining these results with the bounds deduced from (6), we obtain the desired result.

Discriminant analysis. Another interesting application of Theorem 2 is in the context of discriminant analysis for stationary time series (see for example Taniguchi and Kakizawa, 2000, Chapter 7). Suppose that (X_t) is a real Gaussian process with mean $\mu_t = E(X_t)$ which may depend on time t , and a stationary covariance function $\gamma_j = \text{Cov}(X_{t+j}, X_t)$ with $\gamma_0 > 0$. The problem is to identify the mean function μ_t between two possible candidates $\mu_{1,t}$ and $\mu_{2,t}$. Let $X = (X_1, \dots, X_n)'$, $\mu_1 = (\mu_{1,1}, \dots, \mu_{1,n})'$, $\mu_2 = (\mu_{2,1}, \dots, \mu_{2,n})'$, and D the linear discriminant function (LDF),

$$D(X) = (\mu_1 - \mu_2)' \Gamma_n^{-1} X - \frac{1}{2} \mu_1' \Gamma_n^{-1} \mu_1 + \frac{1}{2} \mu_2' \Gamma_n^{-1} \mu_2.$$

X is classified as having mean μ_1 or μ_2 according to whether $D(X) > C$ or $D(X) \leq C$ where C is a constant. The misclassification probabilities are $P(2|1) = \Phi(C/\Delta_n - \Delta_n/2)$ and $P(1|2) = 1 - \Phi(C/\Delta_n + \Delta_n/2)$, where Φ is the cumulative distribution function of the standard normal distribution, and $\Delta_n^2 = (\mu_1 - \mu_2)' \Gamma_n^{-1} (\mu_1 - \mu_2)$ is the Mahalanobis distance. For $C = 0$, $P(2|1) = P(1|2) = \Phi(-\Delta_n/2)$ and these probabilities are decreasing in Δ_n . The LDF is said to be consistent if $\Phi(-\Delta_n/2) \rightarrow 0$ as $n \rightarrow \infty$, which is equivalent to $\Delta_n \rightarrow \infty$ as $n \rightarrow \infty$. Assume that $\gamma_j = \int_I e^{jv} f(v) dv$ where f satisfies (5), and let $\delta = \mu_1 - \mu_2$. We deduce from (6) the following bounds for the Mahalanobis distance,

$$C_2^{-1} \delta' \delta n^{-2 \max(0, d_m)} \leq \Delta_n^2 \leq C_1^{-1} \delta' \delta n^{-2 \min(0, d_1)}. \quad (23)$$

Suppose that $\delta' \delta \sim Cn^\beta$ as $n \rightarrow \infty$ where $C > 0$. We deduce from (23) that the LDF is consistent if $\beta > 2 \max(0, d_m)$, and it is not consistent if $\beta \leq 2 \min(0, d_1)$. Since $d_m < 1/2$, the LDF is consistent for any $\beta \geq 1$. Besides, consider the case where $d_1 > -1/2$ in Theorem 2 as for instance, for an invertible generalized ARFIMA process. Then the LDF is not consistent for any $\beta \leq -1$. Finally, if $\delta' \delta \sim C\beta^n$ as $n \rightarrow \infty$ where $C > 0$ and $\beta > 0$, we deduce from (23) that the LDF is consistent if $\beta > 1$, and is not consistent if $\beta < 1$.

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