Change Points in ARFIMA Models

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The problem of estimating the location of a change in the level of a time series has received considerable attention in the statistical literature, see for example Bai (1994), Wright (1998) and Kuan and Hsu (1998) among others. This work extends the change point estimation methodology put forward by Bai (1994) to ARFIMA processes. The consistency of the least square estimates is derived and their performance is studied by means of Monte Carlo experiments.

1. Change point model

Consider the following model for a shift in the level of a long–memory process $\{y_t\}$:

$$y_t = \mu_1 + \lambda \mathrm{I}(t \geq \tau T) + \epsilon_t, \tag{1}$$

for $t = 1, ..., T$, where $\lambda = \mu_2 - \mu_1$ is the magnitude of the shift, τT is the unknown break time, and ϵ_t is a long–memory ARFIMA model, defined by the discrete-time equation

$$\Phi(B)(1-B)^d \epsilon_t = \Theta(B)\eta_t, \tag{2}$$

for $t = 1, ..., T$, where $d < \frac{1}{2}$, $\{\eta_t\}$ is a white noise sequence, B is the backshift operator $By_t = y_{t-1}$, $\Phi(B)$ and $\Theta(B)$ are polynomials of degrees p and q respectively with no common zeroes and all their roots outside the unit circle, and $(1-B)^d$ is the fractional difference operator. ARFIMA models are generalizations of the ARMA processes, see Beran (1994, p.59). The change point fraction τ may be estimated by means of the Wald statistic

$$W_T(\tau) = \left\{ \sum_{t=1}^{T} (y_t - \bar{y})^2 - \sum_{t=1}^{\tau T} (y_t - \bar{y}_1)^2 - \sum_{t=\tau T}^{T} (y_t - \bar{y}_2)^2 \right\} \hat{\sigma}_\epsilon^{-2} \tag{3}$$

where $\hat{\sigma}_\epsilon^2 = \sum_{t=1}^{T} (y_t - \bar{y})^2/(T-1)$, $\bar{y}_1 = \sum_{t=1}^{\tau T} y_t/(\tau T)$ and $\bar{y}_2 = \sum_{t=\tau T}^{T} y_t/(T - \tau T)$. Then $\hat{\tau} = \mathrm{argmax}_{\tau \in \Delta} W_T(\tau)$, where Δ is a compact subset of $(0, 1)$, and the change point may be estimated by $\hat{k} = \hat{\tau} T$. The weak consistency of the break fraction estimate $\hat{\tau}$ for long–memory processes is established in Theorem 1. Proof of this result is available from the author. (cf. Kuan and Hsu (1998)).

Theorem 1 *For* $d \in (0, \frac{1}{2})$, $\quad |\hat{\tau} - \tau| = T^{-1/2} \lambda^{-1} O_p(T^d)$.

2. Simulations

In order to illustrate the estimation technique previously discussed, several Monte Carlo experiments are carried out. These simulations have 500 repetitions with sample sizes of $T = 200$ and a true trend break at $k_0 = 100$ in the time series:

$$y_t = 1 + \lambda \mathrm{I}(t \geq 100) + \epsilon_t, \tag{4}$$

for $t = 1, ..., 200$, where $\lambda = 2$ and ϵ_t is a fractional noise model: $(1-B)^d \epsilon_t = \eta_t$, $t = 1, ..., 200$ with $\sigma_\eta = 1$. Four values of the long–memory parameter are used, from 0.10 to 0.40. Table 1 displays the results. Notice that as d increases, the precision of the estimate $\hat{k}$ deteriorates.

Table 1. Mean and Standard Deviation of $\hat{k}$

d	0.00	0.10	0.20	0.30	0.40
mean	100.00	99.99	99.54	99.96	99.47
S.D.	1.23	2.02	2.74	8.48	16.55

Since the variance of a long-memory time series generated by (4) with $\sigma_\eta = 1$ changes for different values of d, four Monte Carlo experiments with varying shift magnitude (λ) are carried out. In each case, λ is chosen such that the magnitude of the shift is equal to $2\sigma_\epsilon$. Table 2 displays the results. It can be observed that even though the standard deviations of $\hat{k}$ are smaller than those in Table 1, they increase as d increases.

Table 2. Mean and Standard Deviation of $\hat{k}$

d	0.10	0.20	0.30	0.40
σ_ϵ	1.01	1.05	1.15	1.44
λ	2.01	2.10	2.30	2.88
mean	99.98	99.93	100.49	99.44
S.D.	1.94	2.89	6.50	6.72

3. Conclusions

Some properties of least squares estimates of a shift in long–memory processes are discussed in this paper. One advantage of this methodology is that no distributional assumptions have to be made in order to find the estimates. Another advantage is that the computational algorithm is $O(T)$, allowing for an efficient application of the least squares techniques to large datasets, which are usually the case in a long–memory context.

REFERENCES

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FRENCH RÉSUMÉ

Ce travail présente résultants théoriques et empiriques associés à problemes d'estimation des changes dans les niveaux des modèles ARFIMA.